Interest Rate Dynamics and Commodity Prices[1]

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1. Introduction

Commodity prices are major determinants of exchange rates, tax revenues, the balance of payments, output fluctuations, and inflation around the world (see, e.g., Byrne et al., 2013; Gospodinov and Ng, 2013; Eberhardt and Presbitero, 2021; Peersman, 2022). While some commodity price movements are driven by idiosyncratic shocks to supply and demand, aggregate demand and monetary conditions also play a key role. Alquist et al. (2020) show that up to 80% of the variance of commodity prices is explained by common factors (see also Byrne et al., 2013). Aggregate factors are particularly important when considering the impact of commodities on inflation and exchange rates because such factors induce price comovement in all or many commodities.

Historically, the aggregate factor that has generated most attention is interest rates. For example, Frankel (2008b, 2008c, 2018) has long argued that interest rates are a major driver of comovements in commodity prices, with rising interest rates decreasing commodity prices and falling interest rates increasing them. The main argument relates to "cost of carry:" higher interest rates reduce demand for inventories, which exerts downward pressure on commodity prices. At the same time, it is easy to imagine scenarios where interest rates and commodity prices are *positively* correlated—for example, when high aggregate demand boosts both commodity prices and the cost of borrowing (through credit markets and, potentially, the responses of monetary authorities).

As the previous paragraph suggests, empirical studies of the sign and magnitude of interest rate effects on commodity prices face deep challenges because of the endogeneity and the general equilibrium nature of the mechanisms in question. In particular, interest rate movements are endogenously determined by several macroeconomic variables that also affect commodity markets. Even if we fully control for changes in output and demand, rising commodity prices might themselves trigger a tightening of monetary policy, without

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any change in output (see, e.g., Cody and Mills, 1991). Conversely, pure monetary shocks affect commodity markets through various channels (e.g., speculation, aggregate demand, and supply response) that are hard to disentangle empirically.[2]

These challenges demand a structural model built on firm theoretical foundations that can isolate the direct effect of interest rates on commodity prices through each of the channels listed above. The obvious candidate to provide the necessary structure is the rational expectations competitive storage model developed by Samuelson (1971), Newbery and Stiglitz (1982), Wright and Williams (1982), Scheinkman and Schechtman (1983), Deaton and Laroque (1992, 1996), and Chambers and Bailey (1996), among others.[3] In this model, commodities are assets that also have intrinsic value, separate from future cash flows. The standard version of the model features time-varying production, storage by risk-neutral forward-looking investors, arbitrage constraints, and non-negative carryover (i.e., stocks can only be positive or null). Within the constraints of the model, there is a clear relationship between interest rates, storage, and commodity prices. Fama and French (1987) empirically show that the relationship between the basis, the spread between the futures and spot prices, and interest rates is consistent with the structure of this model.

The main obstacle to applying the standard competitive storage model to the problem at hand is that the model presented in the existing literature has a discount rate that is both time and state-invariant. The source of this shortcoming of the standard model is technical: a constant positive interest rate is central to the traditional proof of the existence and uniqueness of equilibrium prices, as well as to the study of their properties (see, e.g., Deaton and Laroque, 1992, 1996). In particular, positive constant rates are used to obtain contraction mappings over a space of candidate price functions, with the discount factor being the modulus of contraction.

At the same time, relaxing the assumption of constant discounting is essential for any serious analysis of the interactions between interest rates and commodity prices. One reason is that real interest rates do in fact exhibit very large movements over time, as shown in Figure 1.[4] Another is that the exact nature and timing of shocks to supply, demand and interest rates have important implications in terms of sign and magnitude of interest rates effects.

Relaxing the assumption of constant interest rates in the competitive storage model is

[2]For example, a decline in the US interest rate can stimulate both global demand (see, e.g., Ramey, 2016) and firms' incentive to hold inventories (see, e.g., Frankel, 1986, 2008a, 2014), which then increase commodity prices. An increase in interest rates works in the opposite direction.

[3]Samuelson (1971) proved that the equilibrium quantities produced by this model are welfare maximizing for a representative consumer with marginal utility equal to the inverse demand function. Deaton and Laroque (1996) and Cafiero et al. (2011) showed that the model can replicate several important features of the data.

[4]Data source: FRED. The real interest rate is calculated based on the one-year treasury yield and a measure of expected inflation. See Section 4 regarding calculations detail for Figure 1.

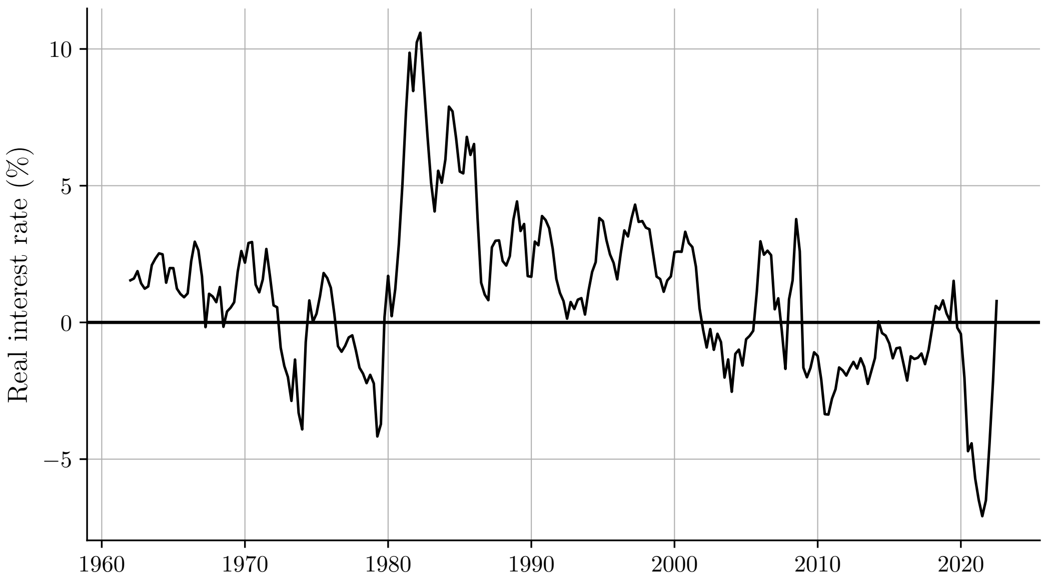

Figure 1 – The US real interest rate over the long run

nontrivial. The main reason is that negative real interest rates cannot be ignored, as is clear from Figure 1. If interest rates can be sufficiently negative for sufficiently long periods of time, then the model will have no finite equilibrium (due to unbounded demand for inventories). Thus, developing a model that can handle realistic calibrations requires accommodating negative yields on risk-free bonds in some states of the world, while providing conditions on these states and the size of the yields such that the model retains a well-defined and unique solution.

In this paper, we begin by extending the competitive storage model to the setting of state-dependent discounting and establishing conditions under which a unique equilibrium price process exists. These conditions allow for both positive and negative discount rates, while also providing a straightforward link between the asymptotic return on risk-free assets with long maturity, the depreciation rate of the commodity in question, and the existence and uniqueness of solutions. Under these conditions, in addition to existence and uniqueness, we show that the equilibrium solution can be computed efficiently on the basis of a globally convergent algorithm and provide a sharp characterization of the continuity and monotonicity properties of the equilibrium objects.

With these results in hand, we turn to an examination of the effect of interest rates on commodity prices from a theoretical and quantitative perspective. We show that, in some settings, interest rates and commodity prices can be positively correlated, such as when shocks that shift up interest rates also increase aggregate demand. Thus, finding clear conditions under which interest rates and commodity prices exhibit negative correlation is nontrivial. Nonetheless, we are able to provide relatively sharp conditions under which negative correlation is realized. In particular, we show that if the exogenous state follows a monotone Markov process that is independent across dimensions and has a non-negative effect on interest rate and commodity output, then interest rates and commodity prices will exhibit negative contemporaneous correlation.[5] A simple example, explored quantitatively in Section 4, is if interest rate follows an autoregressive process and commodity output is iid, then a positive interest rate shock raises the cost of holding stocks, leading to a reduction in stock levels that are sold on the market, the additional market supply depresses current price, which leads to a negative correlation between interest rate and commodity price.

On the quantitative side, we study the impulse response functions (IRFs) of commodity price, inventory, and price volatility to an interest rate shock for a combination of structural parameters typical of commodity markets. As a special application of our theory, we examine the speculative channel—the role of speculators in the physical market whose incentive to hold inventories is affected by interest rate movements—which has been regularly proposed but whose analysis has been neglected thus far. To capture the nonlinear dynamics of the competitive storage model, we follow the approach of Koop et al. (1996), who define the IRFs as state-and-history-dependent random variables. The simulated IRFs show that prices immediately fall after a positive interest rate shock and slowly converge to their long-run value. Moreover, an interest rate increase depresses inventories and increases price volatility. Stocks fall immediately with the shock, but reach their lowest value after a long time, and converge to their long-run average even more slowly than prices. Price volatility tends to follow inventory dynamics: a larger response in inventory causes an inversely larger response in price volatility. Finally, the magnitude and overall pattern of the IRFs depend substantially on the market supply and prevailing interest rate.[6]

Regarding existing literature on interest rates and commodity prices, Jeffrey Frankel has made numerous empirical and theoretical contributions to this topic, focusing on how commodity prices overshoot their long-run target after a shock due to their inherent price flexibility (Frankel and Hardouvelis, 1985; Frankel, 1986, 2008a, 2014). This literature tends to find a negative effect of interest rate increases on commodity prices in the short (Rosa, 2014; Scrimgeour, 2015) and medium run (Anzuini et al., 2013; Harvey et al., 2017).[7] The negative relationship between interest rates and commodity prices has also been found by Christiano et al. (1999) and Bernanke et al. (2005) in other contexts.

[5] The condition we propose is also partly necessary, in the sense that if different exogenous states are contemporaneously correlated, then the trend of comovement could be disrupted and the relationship between interest rates and commodity prices could be reversed. In general, the overall effect could be strengthened or weakened once this independence-across-dimensions condition is violated, in which case a quantitative analysis is necessary.

[6] These results suggest that postulating an invariant effect of monetary shocks under different market supply and interest rates may cause biases in empirical analysis.

[7] An exception is Kilian and Zhou (2022), who find no effect of real interest rate movements on oil prices.

Moreover, interest rates affect not only the level of commodity price, but also their cross-correlation and their volatility as shown by Gruber and Vigfusson (2018). Compared to these studies, the methodology developed here allows for a systematic analysis of the transmission mechanisms. In particular, our quantitative analysis clarifies the role of the speculative channel in the observed negative relationship.

Our work also contributes to the study of the theoretical and empirical properties of the competitive storage model. Theoretically, we derive new results on the existence and uniqueness of a rational expectations equilibrium in a general competitive storage model with state-dependent and time-varying discount factors, which extend the results of Scheinkman and Schechtman (1983), Deaton and Laroque (1992), and Cafiero et al. (2015), and are crucial for studying the dynamic causal effect of these factors on commodity price fluctuations.[8] Moreover, we provide a sharp characterization of the analytical properties of the equilibrium solution, on the basis of which we develop a suitable endogenous grid algorithm that solves the equilibrium objects efficiently and accurately.

Empirically, since Deaton and Laroque (1996), a literature has aimed at analyzing the empirical validity of the storage model (e.g., Cafiero et al., 2011, 2015; Gouel and Legrand, 2022). This literature focuses on idiosyncratic shocks and neglects all shocks to storage costs. In this paper, we make the first attempt to study the role of aggregate shocks on storage costs. We provide the first theoretical analysis of the general conditions under which interest rates are negatively correlated with commodity prices, and give a quantitative analysis of the impact of this aggregate shock on commodity price dynamics through the speculative channel.

The rest of the paper is organized as follows. Section 2 formulates a rational expectations competitive storage model with time-varying discounting and discusses the existence, uniqueness, and computability of the equilibrium solutions. Sections 3 and 4 examine the role of interest rates on commodity prices from a theoretical and quantitative perspective, respectively. Section 5 concludes. Proofs, descriptions of algorithms, and counterexamples can be found in the appendices.

2. Equilibrium Prices

This section formulates the equilibrium problem for the competitive storage model in a time-varying interest rate environment and discusses conditions under which existence and uniqueness of the equilibrium pricing rule hold.

[8] See also Basak and Pavlova (2016) for another storage model with stochastic discount factor but which abstracts from the non-negative constraint on storage.

2.1. The Model

Let $I_t \geq 0$ be the inventory of a given commodity at time t, and let $\delta \geq 0$ be the instantaneous rate of stock deterioration. The cost of storing I_t units of goods from time t to time $t+1$, paid at time t, is kI_t, where $k \geq 0$. Let Y_t be the output of the commodity. Let X_t be the total available supply at time t, which takes values in $\mathsf{X} := [b, \infty)$, where $b \in \mathbb{R}$, and is defined by

$$X_t := \mathrm{e}^{-\delta} I_{t-1} + Y_t. \tag{1}$$

Let $p \colon \mathsf{X} \to \mathbb{R}$ be the inverse demand function. We assume that p is continuous, strictly decreasing, and bounded above.[9] Let P_t be the market price at time t. In the absence of inventory, $P_t = p(Y_t)$. In general, market equilibrium requires that total supply equals total demand (sum of the consumption and the speculation demand), equivalently,

$$X_t = p^{-1}(P_t) + I_t. \tag{2}$$

An immediate implication of (2) is that $P_t \leq p(b)$ and

$$P_t \geq p(X_t), \quad \text{with equality holding when } I_t = 0. \tag{3}$$

Let M_{t+1} be the one-period stochastic discount factor applied by investors at time t. The price process $\{P_t\}$ is restricted by

$$P_t \geq \mathrm{e}^{-\delta} \, \mathbb{E}_t \, M_{t+1} P_{t+1} - k, \quad \text{with equality holding if } I_t > 0 \text{ and } P_t < p(b). \tag{4}$$

In other words, per-unit expected discounted returns from storing the commodity over one period cannot exceed the per-unit cost of taking that position.

Combining (3) and (4) yields[10]

$$P_t = \min \left\{ \max \left\{ \mathrm{e}^{-\delta} \, \mathbb{E}_t \, M_{t+1} P_{t+1} - k, \; p(X_t) \right\}, \; p(b) \right\}. \tag{5}$$

Both $\{M_t\}$ and $\{Y_t\}$ are exogenous, obeying

$$M_t = m(Z_t, \varepsilon_t) \quad \text{and} \quad Y_t = y(Z_t, \eta_t), \tag{6}$$

where m and y are measurable functions satisfying $m \geq 0$ and $y \geq b$, $\{Z_t\}$ is a time-homogeneous irreducible Markov chain (possibly multi-dimensional) taking values in a finite set Z, and the innovations $\{\varepsilon_t\}$ and $\{\eta_t\}$ are iid and mutually independent.

[9] We impose an upper bound to simplify exposition. In Appendix A we show that unbounded demand functions can also be treated and theory below still holds.

[10] The minimization over $p(b)$ in (5) is required due to the generic stochastic discounting setup. As can be seen below, our theory allows for large and highly persistent discounting process (e.g., arbitrarily long sequences of negative low interest rates under risk neutrality), in which case $\mathrm{e}^{-\delta} \, \mathbb{E}_t \, M_{t+1} > 1$ with positive probability, thus the marginal reward of speculation, $\mathrm{e}^{-\delta} \, \mathbb{E}_t \, M_{t+1} P_{t+1}$, can be larger than $p(b)$. The extra minimization operation is then required to meet the equilibrium condition $P_t \leq p(b)$.

Example 2.1. The setup in (6) is very general and allows us to model both correlated and uncorrelated $\{M_t, Y_t\}$ processes. In particular, it does not impose that $\{M_t\}$ and $\{Y_t\}$ are driven by a *common* Markov process, nor does it restrict that they are mutually dependent. Consider for example $Z_t = (Z_{1t}, Z_{2t})$, where $\{Z_{1t}\}$ and $\{Z_{2t}\}$ are mutually independent, possibly multi-dimensional Markov processes, and $M_t = m(Z_{1t}, \varepsilon_t)$ and $Y_t = y(Z_{2t}, \eta_t)$. In this case, $\{M_t\}$ and $\{Y_t\}$ are mutually independent, although they are autocorrelated. If in addition $\{Z_{1t}\}$ (resp., $\{Z_{2t}\}$) is iid or does not exist, then $\{M_t\}$ (resp., $\{Y_t\}$) is iid. Obviously, these are all special cases of (6). More examples are given in Section 3 below.

Below, the next period value of a random variable X is denoted by $\hat{X}$. In addition, we define $\mathbb{E}_z := \mathbb{E}(\cdot \mid Z = z)$ and assume throughout that

$$e^{-\delta}\, \mathbb{E}_z\, \hat{M} p(\hat{Y}) - k > 0 \quad \text{for all} \quad z \in \mathsf{Z}. \tag{7}$$

In other words, the present market value of future output covers the cost of storage.

2.2. Discounting

To discuss conditions under which price equilibria exist, we need to jointly restrict discounting and depreciation. To this end, we introduce the quantity[11]

$$\kappa(M) := \lim_{n\to\infty} \frac{-\ln q_n}{n} \qquad \text{where} \qquad q_n := \mathbb{E} \prod_{t=1}^{n} M_t. \tag{8}$$

To interpret $\kappa(M)$, note that, in this economy, $q_n(z) := \mathbb{E}_z \prod_{t=1}^{n} M_t$ is the state z price of a strip bond with maturity n. Since $\{Z_t\}$ is irreducible, initial conditions do not determine long run outcomes, so $q_n(z)$ is approximately constant at q_n defined in (8) when n is large. As a result, we can interpret $\kappa(M)$ as the asymptotic yield on risk-free zero-coupon bonds as maturity increases without limit.

In Lemma A.1 of Appendix A, we provide a numerical method for calculating $\kappa(M)$ by connecting it to the spectral radius of a discount operator.
Assumption 2.1. $\kappa(M) + \delta > 0$.

Assumption 2.1 is analogous to the classical condition $r + \delta > 0$ found in constant interest rate environment of Deaton and Laroque (1996) and many other studies.[12] In the more general setting we consider, Assumption 2.1 ensures sufficient discounting, adjusted by the depreciation rate, to generate finite prices in the forward-looking recursion (5), while still allowing for arbitrarily long sequences of negative yields in realized time series.

[11] Here and below, expectation without a subscript refers to the stationary process, where Z_0 follows the (necessarily unique) stationary distribution.
[12] In the model with constant risk-free rate r, the discount rate M_t is $1/(1+r)$ at each t, so, by the definition in (8), we have $\kappa(M) = \lim_{n\to\infty} n \ln(1 + r)/n = \ln(1 + r) \approx r$.

2.3. Equilibrium

We take (X_t, Z_t) as the state vector, taking values in $\mathsf{S} := \mathsf{X} \times \mathsf{Z}$. To ensure that the equilibrium prices are non-negative, we assume free disposal as in Cafiero et al. (2015). Conjecturing that a stationary rational expectations equilibrium exists and satisfies (5), an *equilibrium pricing rule* is defined as a function $f^* \colon \mathsf{S} \to \mathbb{R}_+$ satisfying

$$f^*(X_t, Z_t) = \min\left\{\max\left\{e^{-\delta}\, \mathbb{E}_t\, M_{t+1} f^*(X_{t+1}, Z_{t+1}) - k,\ p(X_t)\right\},\ p(b)\right\}$$

with probability one for all t, where X_{t+1} is defined by (1) and, recognizing free disposal, storage therein is determined by $I_t = i^*(X_t, Z_t)$, where $i^* : \mathsf{S} \to \mathbb{R}_+$ is the *equilibrium storage rule*[13]

$$i^*(x, z) := \begin{cases} x - p^{-1}[f^*(x, z)], & \text{if } x < x^*(z) \\ x^*(z) - p^{-1}(0), & \text{if } x \geq x^*(z) \end{cases} \tag{9}$$

with

$$x^*(z) := \inf\left\{x \in \mathsf{X} : f^*(x, z) = 0\right\}.$$

Let $\mathscr{C}$ be the space of bounded, continuous, and non-negative functions f on S such that $f(x, z)$ is decreasing in x, and $f(x, z) \geq p(x)$ for all (x, z) in S. Given an equilibrium pricing rule f^*, let

$$\bar{p}(z) := \min\left\{e^{-\delta}\, \mathbb{E}_z\, \hat{M} f^*(\hat{Y}, \hat{Z}) - k,\ p(b)\right\}.$$

The next theorem provides conditions under which the equilibrium pricing rule exists, is uniquely defined, and gives a sharp characterization of its analytical properties.

Theorem 2.1. *If Assumption 2.1 holds, then there exists a unique equilibrium pricing rule f^* in the function space $\mathscr{C}$. Furthermore,*

(i) $f^*(x, z) = p(x)$ *if and only if* $x \leq p^{-1}[\bar{p}(z)]$,
(ii) $f^*(x, z) > \max\{p(x), 0\}$ *if and only if* $p^{-1}[\bar{p}(z)] < x < x^*(z)$,
(iii) $f^*(x, z) = 0$ *if and only if* $x \geq x^*(z)$, *and*
(iv) $f^*(x, z)$ *is strictly decreasing in x when it is strictly positive and* $e^{-\delta}\, \mathbb{E}_z\, \hat{M} < 1$.

In Appendix A, we show that the equilibrium pricing rule is the unique fixed point of an operator defined by the equilibrium conditions and can be solved for via successive approximation. In Appendix E, we design a suitable endogenous grid algorithm based on the theory above, which allows us to solve the equilibrium objects accurately and efficiently.

The next result states properties of the equilibrium storage rule.

Proposition 2.1. *If Assumption 2.1 holds, then the equilibrium storage rule $i^*(x, z)$ is increasing in x and continuous. Furthermore,*

[13] Throughout, we adopt the usual convention that $\inf \emptyset = \infty$.

(i) $i^*(x,z) = 0$ *if and only if* $x \leq p^{-1}[\bar{p}(z)]$,

(ii) $0 < i^*(x,z) < x^*(z) - p^{-1}(0)$ *if and only if* $p^{-1}[\bar{p}(z)] < x < x^*(z)$,

(iii) $i^*(x,z) = x^*(z) - p^{-1}(0)$ *if and only if* $x \geq x^*(z)$, *and*

(iv) $i^*(x,z)$ *is strictly increasing in* x *if* $p^{-1}[\bar{p}(z)] < x < x^*(z)$ *and* $\mathrm{e}^{-\delta}\,\mathbb{E}_z \hat{M} < 1$.

Proposition 2.1 indicates that speculators hold inventories if and only if the market value of the total available supply $p(x)$ is below the decision threshold $\bar{p}(z)$. Otherwise, selling all commodities at hand is optimal, in which case the equilibrium price is $f^*(x,z) = p(x)$. Properties of the equilibrium price and storage are illustrated in Figure 2 under a linear demand function. The equilibrium rules are sketched for a given exogenous state z.

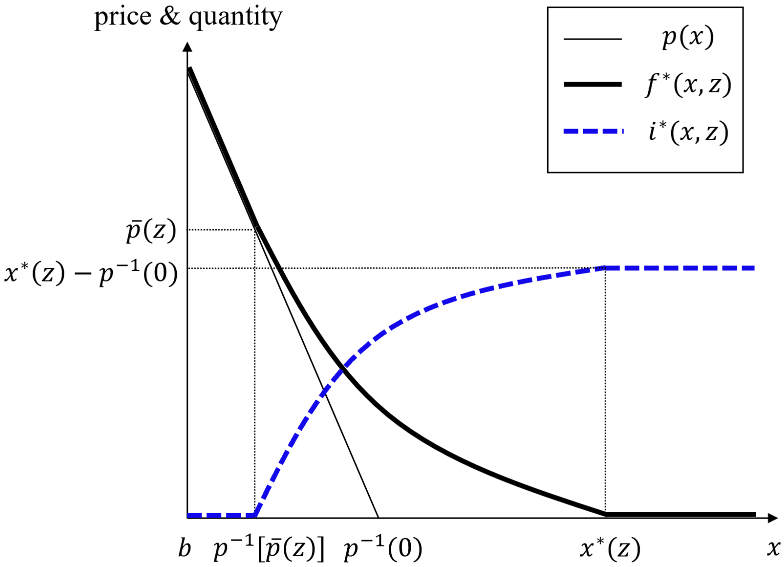

Figure 2 – Illustration of the equilibrium price and the equilibrium storage

3. Interest Rates and Prices: Theoretical Results

This section inspects the relationship between interest rates and commodity prices from a theoretical perspective. To this end, we assume that speculators discount future payoffs according to market prices. In other words,

$$M_t = \frac{1}{R_t}, \quad \text{where} \quad R_t := r(Z_t, \varepsilon_t).$$

Hence r is a real-valued non-negative Borel measurable function of the state process and innovation ε. The process $\{R_t\}$ is interpreted as (and will later be estimated to match) the gross real interest rate on risk-free bonds. We therefore preserve the risk-neutrality assumption of the standard competitive storage model, while allowing the risk-free rate to be state-dependent. Throughout this section, we assume that the assumptions of Section 2 are valid.

3.1. Ordered Prices

We first state a monotonicity property concerning interest rates and prices. To this end, we take $\{R_{it}\}$ to be the interest rate process for economy $i \in \{1, 2\}$. In addition, let f_i^* and $\{P_{it}\}$ be the equilibrium pricing rule and the price process corresponding to $\{R_{it}\}$.

Proposition 3.1. *If $R_{2t} \leq R_{1t}$ with probability one for all $t \geq 0$, then $f_1^* \leq f_2^*$ and $P_{1t} \leq P_{2t}$ with probability one for all t.*

The intuition is straightforward. Seen from the speculative channel, lower interest rates reduce the opportunity cost of storage. Lower storage costs encourage a build-up of inventories. Higher demand for inventories induces higher prices.

Proposition 3.1 has limited implications because it concerns variations in interest rates that are uniformly ordered over time. The results in the next section allow us to consider more refined variations.

3.2. Correlations

Below we explore general conditions under which interest rates and commodity prices are negatively correlated and discuss their necessity. As a first step, we state a key finding concerning monotonicity of equilibrium objects with respect to the exogenous states.

Proposition 3.2. *If $r(z, \varepsilon)$ and $y(z, \eta)$ are nondecreasing in z, and $\{Z_t\}$ is a monotone Markov process,*[14] *then the equilibrium pricing rule $f^*(x, z)$, the equilibrium inventory $i^*(x, z)$, and the decision threshold $\bar{p}(z)$ are all decreasing in z.*

The intuition is as follows: If (i) a higher Z_t shifts up the distribution of Z_{t+1} in terms of first-order stochastic dominance and (ii) interest rates and output are both nondecreasing in this state variable, then a high Z_t today tends to generate both sustained high output and more impatient speculators in the future. The former boosts aggregate supply, while the latter reduces the incentive for holding inventories, which in turn reduces aggregate demand. As a result, both inventories and prices are lower.

The assumptions of Proposition 3.2 do not restrict R_t and Y_t to be strictly increasing in Z_t, nor do they impose that R_t and Y_t are driven by a common factor. In particular, the second assumption concerning monotone Markov process is standard (see Appendix B for sufficient conditions). Below, we discuss the first assumption through examples.

Example 3.1. If $\{R_t\}$ and $\{Y_t\}$ are iid and mutually independent, then we can set $Z_t \equiv 0$, $\varepsilon_t = R_t$ and $\eta_t = Y_t$, in which case $r(z, \varepsilon) = \varepsilon$ and $y(z, \eta) = \eta$. Hence the first two assumptions of Proposition 3.2 hold.

Example 3.2. If $\{R_t\}$ and $\{Y_t\}$ are autocorrelated and mutually independent, then we can write Z_t as $Z_t = (Z_{1t}, Z_{2t})$, where $\{Z_{1t}\}$ and $\{Z_{2t}\}$ are mutually independent, possibly

[14] Here monotonicity is defined in terms of first-order stochastic dominance. See Appendix B for its formal definition.

multi-dimensional Markov chains, and $R_t = r(Z_{1t}, \varepsilon_t)$ and $Y_t = y(Z_{2t}, \eta_t)$. In this case, the first assumption of Proposition 3.2 holds as long as r is nondecreasing in Z_{1t} and y is nondecreasing in Z_{2t}.

Example 3.3. If $\{R_t\}$ and $\{Y_t\}$ are finite Markov processes, then we can set $\varepsilon_t = \eta_t \equiv 0$ and define $Z_t = (R_t, Y_t)$, in which case the first assumption of Proposition 3.2 holds automatically, while the second assumption holds as long as $\{R_t\}$ and $\{Y_t\}$ are monotone and non-negatively correlated Markov processes.

We can now state our main result concerning correlation. In doing so, we suppose that $Z_t = (Z_{1t}, \dots, Z_{nt})$ takes values in $\mathbb{R}^n$.

Proposition 3.3. *If the conditions of Proposition 3.2 hold and $\{Z_{1t}, \dots, Z_{nt}\}$ are independent for each fixed t, then*

$$\mathrm{Cov}_{t-1}(P_t, R_t) \leq 0 \quad \textit{for all } t \in \mathbb{N}.$$

As Proposition 3.2 illustrates, when interest rates and output are both positively affected by the monotone exogenous state process, commodity prices will be negatively affected by the exogenous state. Therefore, there is a trend of comovement (in opposite directions) between commodity price and interest rate, resulting in a negative correlation. The proof of Proposition 3.3 relies on the Fortuin–Kasteleyn–Ginibre inequality.

Remark 3.1. The extra independence-across-dimensions condition in Proposition 3.3 cannot be omitted. In Appendix C, we provide examples showing that if $\{Z_{1t}, \dots, Z_{nt}\}$ are positively or negatively correlated for some $t \in \mathbb{N}$, then interest rates and prices can be positively correlated even under elementary scenarios. This is because the contemporaneous correlation between different dimensions of Z_t could disrupt the trend of comovement of interest rates and commodity prices. In general, a contemporaneous correlation of different dimensions of Z_t can either strengthen or weaken the impact of interest rates on commodity prices, yielding rich model dynamics.

Remark 3.2. In Appendix B, we show that Proposition 3.3 can be extended to the general setting of Section 2, where agents are not necessarily risk neutral. In other words, $\mathrm{Cov}_{t-1}(P_t, M_t) \geq 0$ holds with or without the assumption of risk neutrality.

Example 3.4. (The Speculative Channel). In applications $\{R_t\}$ typically follows a Markov process, while $\{Y_t\}$ represents a sequence of supply shocks (e.g., harvest failures, conflicts around oil production sites, so on), which is iid and less likely to be affected by the monetary conditions (see, e.g., Deaton and Laroque, 1992; Cafiero et al., 2015). Hence, $\{R_t\}$ and $\{Y_t\}$ are mutually independent. In this case, all the effects of interest rates on commodity prices transit through commodity speculation. By letting $Z_t = R_t$, $\varepsilon_t \equiv 0$ and $\eta_t = Y_t$, we have $r(z, \varepsilon) = z$ and $y(z, \eta) = \eta$. Hence, all the assumptions of Proposition 3.3 hold as long as $\{R_t\}$ is a finite monotone Markov process (e.g., a discrete version of a positively correlated AR(1) process) and Assumption 2.1 holds (see the next section). In this case, Proposition 3.3 implies that interest rates are negatively correlated with commodity prices, which matches the empirical results of Frankel (1986, 2008a, 2014).

Example 3.5. (The Global Demand Channel). Since the output of the commodity, Y_t, enters linearly in total availability, it can be redefined as a linear combination of two shocks: $Y_t = Y_t^S - Y_t^D$, where Y_t^S is the supply shock and Y_t^D is the demand shock. Hence Y_t can be interpreted as a net supply shock. There is widespread evidence that both types of shocks matter in commodity markets, albeit with relative importance depending on the commodities (see, e.g., Kilian, 2009; Gouel and Legrand, 2022). Unlike supply shocks, demand shocks are likely to be affected by monetary policies. Since interest rates affect global demand (Ramey, 2016), an interest rate shock leads to an aggregate demand shock that affects all commodities.[15] If interest rates follow a Markov process it implies that $Z_t = (R_t, Z_{2t})$ and $Y_t = y(Z_{2t}, \eta_t)$, where $\{Z_{2t}\}$ is a Markov process that is correlated with $\{R_t\}$. Hence, Z_t is contemporaneously correlated and the independence-across-dimensions condition of Proposition 3.3 fails. However, the theory of Section 2 still applies and can be used to quantify the impact of interest rates on commodity prices.

4. Quantitative Analysis

As one application of our theory, we study the impact of interest rates on commodity prices through the speculative channel. To this end, we shut down the other channels, such as global demand, and use a stylized model that requires a minimum number of parameters to characterize its behavior.[16] We calibrate the model to quarterly setting in order to limit the number of state variables.[17]

The main takeaways from this section are fourfold. First, impulse response functions (IRFs) show that prices fall immediately after a positive interest rate shock and slowly converge to their long-run average. Second, inventories fall more slowly after the shock and converge back to their long-run average even more slowly than prices. Third, price volatility is sensitive to inventory dynamics: a larger response in inventory tends to generate an inversely larger response in price volatility. Fourth, the strength of these IRFs is highly state-dependent, being stronger for high availabilities and/or low interest rates.

[15]Even if the demand shock is an aggregate shock common to all commodities, its effect is likely different depending on the commodities. Demand for food commodities might be little sensitive to GDP shocks due to their low income elasticity. However, metals and energy commodities could be more responsive to global demand shocks since they are used substantially as intermediate inputs.

[16]Gouel and Legrand (2022) show that fitting most moments of a commodity market requires a rich storage model with supply reaction, autocorrelated shocks, and news shocks. Since most of these elements are specific to each commodity market and orthogonal to the question studied here, we abstract for them and focus on a model with only 2 free parameters.

[17]A monthly real interest rate process requires a rich autoregressive structure, introducing many lags.

4.1. Specification

For simulating the model, we adopt a linear demand function

$$p(x) = \bar{p}\left[1 + (x/\mu_Y - 1)/\lambda\right], \tag{10}$$

where $\bar{p} > 0$ is the steady-state price,[18] $\mu_Y > 0$ is the mean of the commodity output process (so also the steady-state consumption level), and $\lambda < 0$ is the price elasticity of demand. We assume that all storage costs are related to depreciation (i.e., $k = 0$ and $\delta \geq 0$). As Gouel and Legrand (2022) show, when calibrated to represent the same proportion of the steady-state price, these two types of storage costs have indistinguishable effects on price moments, so focusing only on one of the two involves no loss of generality.

In addition, we assume that the interest rate follows the first order autoregressive process

$$R_t = \mu_R(1-\rho_R) + \rho_R R_{t-1} + \sigma_R\sqrt{1-\rho_R^2}\,\varepsilon_t^R, \quad \{\varepsilon_t^R\} \overset{\text{iid}}{\sim} N(0,1), \tag{11}$$

and that commodity output, $\{Y_t\}$, follows a truncated normal distribution with mean μ_Y, standard deviation $\mu_Y\sigma_Y$, truncated at 5 standard deviations. The truncation of the distribution (also adopted, inter alia, in Deaton and Laroque, 1992) ensures a lower bound for commodity output and total available supply. This set up is a very special case of the theoretical framework established in Section 3, with $\varepsilon_t \equiv 0$, $\eta_t = Y_t$, $Z_t = R_t$, $r(Z_t, \varepsilon_t) = Z_t = R_t$ and $y(Z_t, \eta_t) = \eta_t = Y_t$. In this setting, $r(z, \varepsilon)$ and $y(z, \eta)$ are increasing in z. Below we estimate the interest rate process and show that $\rho_R > 0$ (implying that $\{R_t\}$ is a monotone Markov process) and that the discount condition in Assumption 2.1 holds. Hence, all the statements of Theorem 2.1 and Propositions 2.1 and 3.2 are valid. Since we have assumed that $\{R_t\}$ is independent of $\{Y_t\}$ to focus explicitly on the speculative channel, the assumptions (and thus conclusions) of Proposition 3.3 also hold.

This choice of parameterization limits the free parameters that matter in the analysis of price movements to δ and λ. Indeed, the interest rate process is estimated on observations, and in this setup in which we will only analyze price movements, adjusting the intercept and slope of the demand function is equivalent to adjusting the mean and variance of the output process (see the proof in Appendix D, which is a generalization of Proposition 1 of Deaton and Laroque, 1996). In addition, we can normalize $\bar{p}$ and μ_Y to any values since their effect is only to set the average price and quantity levels. To clarify the interpretation of the simulations, we normalize $\bar{p}$ and μ_Y to 1, and σ_Y to 0.05.[19]

To calibrate the real interest rate process, we use the one-year treasury yield and a measure of expected inflation. We use the one-year rather than the 3-month treasury yield because

[18] If not otherwise specified, we designate by steady state the equilibrium in the absence of shocks when the agents do not expect any shocks.

[19] According to Gouel and Legrand (2022), a coefficient of variation of 5% for the net-supply shock is slightly above the total shock (demand plus supply) affecting the aggregate crop market of maize, rice, soybeans, and wheat, but below the shocks affecting each of these markets individually.

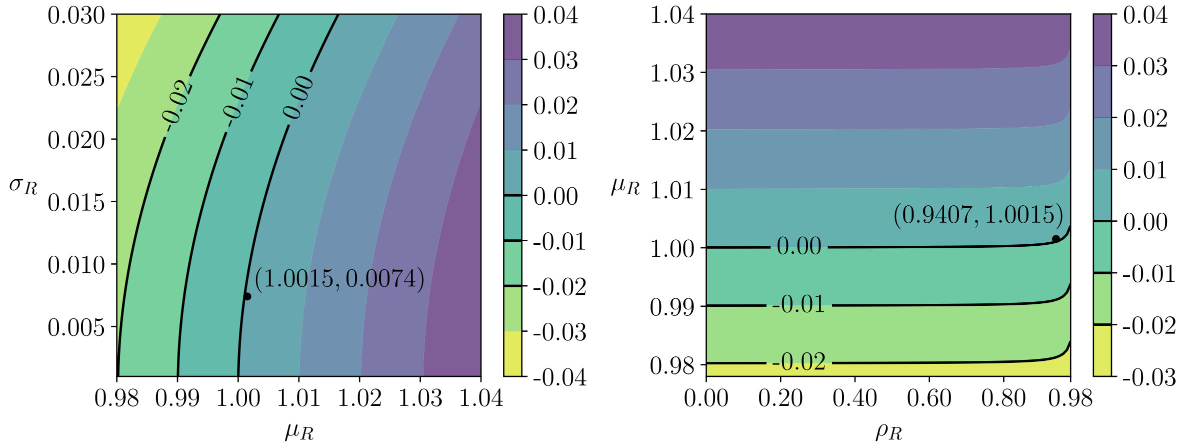

Figure 3 – $\kappa(M)$ values under different $(\mu_R, \rho_R, \sigma_R)$

the latter would lead to lower average real interest rates. Adopting a slightly higher interest rate than the risk-free 3-month rate but with similar dynamics allows us to represent the fact that, in reality, speculators pay a premium above the risk-free rate (also captured here by δ). Expected inflation is calculated through an autoregressive model estimated on a 30-year window prior to the year of interest to account for changes in the dynamics of inflation.[20] This is the real interest rate represented in Figure 1. After converting the annual rate into a quarterly rate, the maximum likelihood estimation of (11) yields $\mu_R = 1.0015$, $\rho_R = 0.9407$, and $\sigma_R = 0.0074$.

These results imply that the stationary mean of the net real interest rate process is about 0.15% (corresponding to an annual rate of 0.6%), with unconditional standard deviation slightly below 1%. In reality speculators may face higher interest rates than the risk-free rates we consider here, but any constant spread above the risk-free rates can be captured in our model by δ, so when interpreting the values of δ in what follows, it should be kept in mind that δ represents at the same time storage costs, spread above risk-free rates, and also any long-run trend in commodity prices.[21] We discretize the interest rate process (11) into an N-state Markov chain using the method of Tauchen (1986). To achieve high precision we set $N = 101$ in the applications.

Our first step is to verify Assumption 2.1, which requires $\kappa(M) > -\delta$. Figure 3 plots $\kappa(M)$ calculated at different $(\mu_R, \rho_R, \sigma_R)$ values.[22] In the left panel, we fix ρ_R at its estimated value and create a contour plot of $\kappa(M)$ for (μ_R, σ_R). In the right panel, we fix σ_R at its estimated value and plot $\kappa(M)$ as a function of (ρ_R, μ_R). The figure shows that $\kappa(M)$ is increasing in μ_R and decreasing in σ_R and ρ_R. In general, $\kappa(M) > -\delta$ fails only when μ_R is sufficiently low, or when ρ_R or σ_R is very large. The black solid curves represent the thresholds at which $\kappa(M) = -0.02, -0.01, 0$, respectively. Clearly, Assumption 2.1 holds

[20] Using lagged inflation or ex-post inflation would lead to a similar real interest rate process.
[21] See Bobenrieth et al. (2021) for an analysis of the role of commodity price trends in the storage model.
[22] The method for computing $\kappa(M)$ is described in Lemma A.1 of Appendix A.

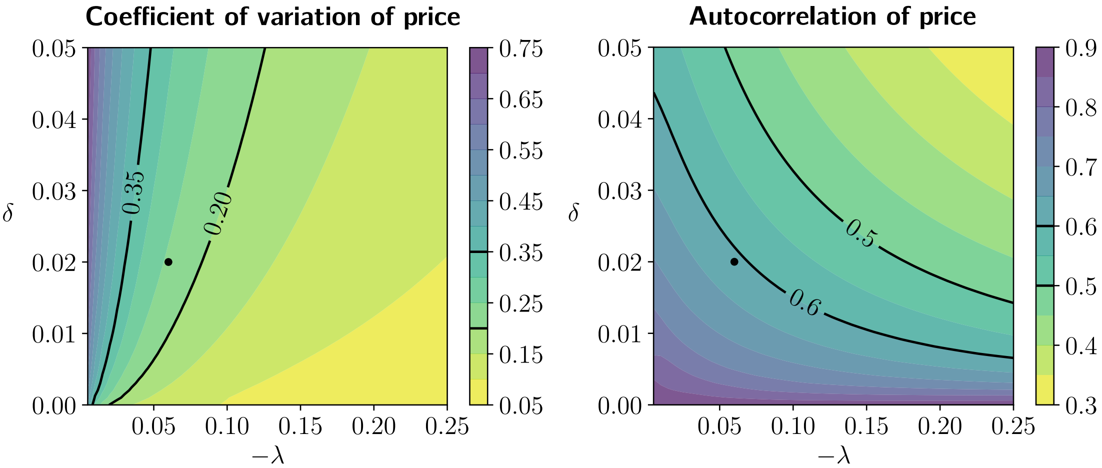

Figure 4 – Unconditional moments (dot point corresponds to $\delta = 0.02$ and $\lambda = -0.06$)

at the estimated $(\mu_R, \rho_R, \sigma_R)$ values even when $\delta = 0$.

Having verified Assumption 2.1, we solve for the equilibrium pricing rule using the following methods. Expectation terms are replaced by simple sums using the interest rate Markov chain and a 7-point Gaussian quadrature for the output process. Starting from a guessed initial solution, the pricing rule is found by iterating with the equilibrium price operator, which is globally convergent by Theorem A.1 in the appendix. To maximize both accuracy and efficiency when iterating, we apply a suitably modified version of the endogenous grid method (Carroll, 2006).[23] Details of the algorithm and computation are given in Appendix E.

4.2. Experiments

Unless otherwise specified, results are presented assuming $\delta = 0.02$ and $\lambda = -0.06$. This combination of parameters leads to the following price moments on the asymptotic distribution: a coefficient of variation of 24%, a first-order autocorrelation of 0.61, and a skewness of 2.9. In order to clarify how these parameters influence price moments, Figure 4 presents the contour plots of coefficient of variation and autocorrelation for a range of values for δ and λ. The coefficient of variation is primarily determined by demand elasticity, and secondarily by storage costs. The autocorrelation of price is jointly determined by both parameters. An increase in the absolute value of either of them will lead to a lower autocorrelation.

Since the Markov process we adopt here is symmetric around the mean, the main quantitative implications of time-varying interest rates are seen in conditional moments, which

[23] When applying the endogenous grid method for time iteration, we use a 100-point exponential grid for I_t in the range of $[0, 2]$ with median value 0.5. Function approximation is implemented via linear interpolation. We terminate the iteration process at precision 10^{-4}.

we explore below with impulse response functions (IRFs). To capture the nonlinear dynamics generated by the storage model, we follow Koop et al. (1996) and define IRFs as state-and-history-dependent random variables. We calculate the IRFs to a 25 bp interest rate impulse (i.e., a 0.25% increase in the real interest rate). All IRFs represent percentage deviation from the benchmark simulation. Detailed discussion of the algorithm and computation are left to Appendix E.

Figure 5 shows the IRFs calculated at the stationary mean of (X_{t-1}, R_{t-1}). The IRFs for prices show an immediate price decrease followed by a gradual convergence to the long-run average over 2 to 4 years. A 25 bp increase in the real interest rate decreases prices at most by -1.62% for the lowest storage costs. The effects decrease with storage costs: the higher the storage costs, the lower the initial decline and the faster the convergence. The cheaper the physical storage costs, the more important the opportunity costs are and so the more interest rate variations weight on storage behavior and prices. Similarly, the effects strengthen the more inelastic is the demand function, because prices react much more to the sales of stock with a more inelastic demand. All combinations of parameters are considered in Figure 6, which displays the contemporaneous impulse of prices to a 25 bp increase in real rates. The figure shows that combinations of low storage costs and inelastic demand can lead to price decreases in excess of 2%.

Figure 5 also shows the IRFs for inventories, allowing us to clarify the mechanisms behind the price decrease. Unlike prices, inventories reach their lowest value more than a year after the shock, and even after 4 years they have not returned to their long-run values. When the interest rate increases, speculators tend to dispose of their costlier stocks. This selling decreases current prices because of the added market supply, mitigating sale of stock relative to constant prices. In subsequent periods, stocks are excessive given the persistence of opportunity costs and speculators continue selling them. Again, the price decrease mitigates the sale. After more than a year of this dynamic, with increasingly smaller quantities sold from inventories, interest rates have decreased, easing the cost pressure to sell inventories, and prices are lower than their long-run values with expectations of converging back. So stock accumulation gradually increases with lower interest rates. Since the stock accumulation is slow in our example, prices converge to their long-run values from below without overshooting (although overshooting is possible). After 4 years, prices have converged to their long-run values, so the only driver of stock increase is the convergence of interest rate to its long-run average.

Stock levels present a different sensitivity to storage costs and demand elasticity. Stocks decrease more with an increase in interest rate when storage costs are lower because, in such settings, stock levels are higher on average and more sensitive to variation in the opportunity costs. Stocks decrease less for a more inelastic demand. This is explained by the fact that with a more inelastic demand, a small increase of sales from inventory depresses prices much more, limiting the incentives to sell too much from the stocks.

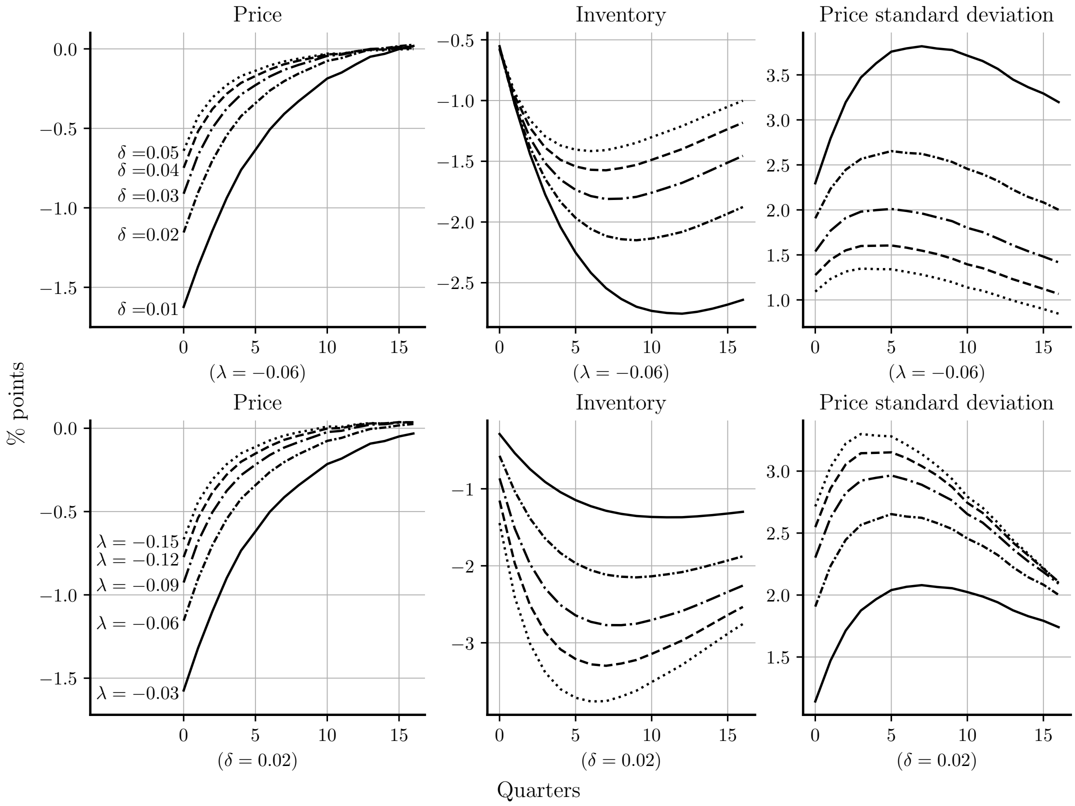

Figure 5 – IRFs for a 25 bp real interest rate shock under different parameter setups (fixing X_{t-1} and R_{t-1} at the stationary mean)

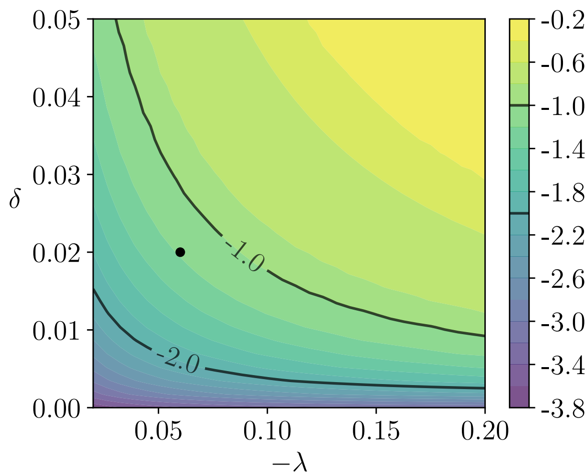

Figure 6 – Contour plot of contemporaneous IRFs (in percentage points) to a 25 bp interest rate shock for various parameters, fixing X_{t-1} and R_{t-1} at the stationary mean. (The dot point represents $\delta = 0.02$ and $\lambda = -0.06$)

The right panels of Figure 5 display the IRFs for price volatility, namely, the conditionally expected standard deviation of price.[24] It shows that price volatility mostly follows stock dynamics with a peak attained after a year and an incomplete convergence after 4 years. A larger response in storage generates an oppositely larger response in price volatility. These results match the empirical results of Gruber and Vigfusson (2018) who show that higher interest rates imply higher price volatility.

To explore the sensitivity of IRFs to states, Figure 7 draws the IRFs calculated for different realized values of (X_{t-1}, R_{t-1}). We use (X^p_{t-1}, R^p_{t-1}) to denote the percentile points of the realized (X_{t-1}, R_{t-1}) states on the stationary distribution. The top left panel shows that price responses are stronger when availability becomes larger. The immediate responses of price are respectively 1.72 and 2.17 times larger when availability increases from the 25% percentile to the 75% and 95% percentiles. This is because when availability is lower, inventory tends to be lower (Proposition 2.1), hence there is less room for stock adjustment and prices react much less in response to the interest rate shock. This intuition is verified by the top middle panel, which shows that a higher availability causes stock decumulation to last longer, yielding a larger decline in inventory in the medium to long run (in spite of a slightly lower immediate decline).

The bottom left panel in Figure 7 shows that price responses to a 25 bp interest rate shock tend to be slightly larger when interest rates are relatively lower. The overall trend of price and storage IRFs in the bottom panels of Figure 7 is consistent with our theory (Proposition 3.2), which shows that under lower interest rates, prices and inventories are in general higher and therefore more sensitive to variations in opportunity costs.

Same to the previous cases, the right panels of Figure 7 show that the dynamics of price volatility are highly consistent with the inventory dynamics, with a larger response in speculative storage causing an oppositely larger response in price volatility.

5. Conclusion

This paper extends the classical competitive storage model to the setting where interest rates are time-varying. We developed a unified theory of how interest rates and other aggregate factors affect commodity prices. We proposed readily verifiable conditions under which a unique equilibrium solution exists and can be efficiently computed. These conditions have natural interpretation in terms of the asymptotic yield on long maturity risk-free assets. We also provided a sharp characterization of the analytical properties of the equilibrium objects, and developed an efficient solution algorithm.

Within this framework, we studied the dynamic causal effect of interest rates on commodity prices. On the theoretical side, we established conditions under which interest rates

[24] Mathematically, the price volatility is defined here as the square root of the conditional variance: $\sqrt{\mathbb{E}_{t-1}[f^*(X_t, Z_t)]^2 - [\mathbb{E}_{t-1} f^*(X_t, Z_t)]^2}$.

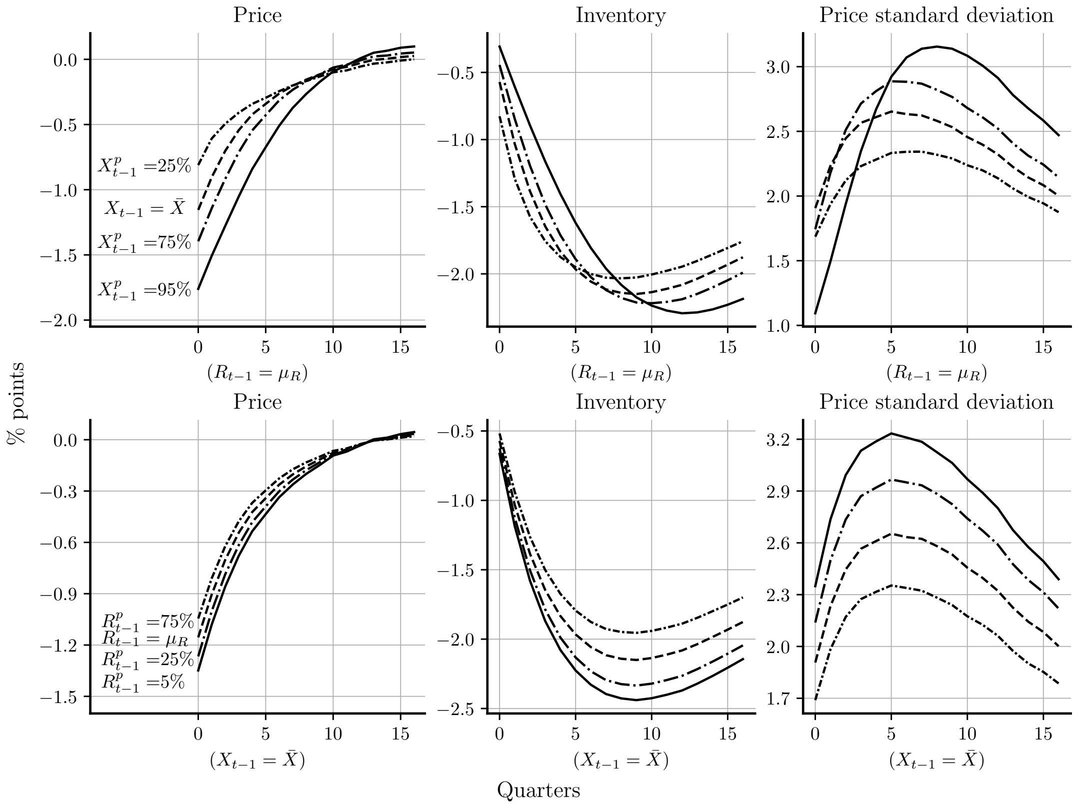

Figure 7 – IRFs for a 25 bp real interest rate shock conditional on different states

and commodity prices are negatively correlated. On the quantitative side, we applied our theory to examine the impact through the speculative channel. Impulse response analysis showed that interest rates have a nontrivial and persistent negative effect on commodity prices in most empirically relevant settings. Moreover, the magnitude of the effect depends substantially on the prevailing market supply and interest rate regime.

The quantitative application in the current paper focuses on the speculative channel. Exploring (i) the impact of interest rates on commodity prices through various other channels (such as the global demand channel), and (ii) the impact of more sophisticated stochastic discount factors (as found, for example, in Schorfheide et al., 2018) and risky returns are equally important. While these topics are beyond the scope of the current paper, the theory presented here lays a solid foundation for new work along these lines.

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Appendix

A. Proof of Section 2 Results

Here and in the remainder of the appendix, we let Φ be the probability transition matrix of $\{Z_t\}$. In particular, $\Phi(z, \hat{z})$ denotes the probability of transitioning from z to $\hat{z}$ in one step. Recall M_t defined in (6). We denote $\mathbb{E}_z := \mathbb{E}(\cdot \mid Z = z)$ and $\mathbb{E}_{\hat{z}} := \mathbb{E}(\cdot \mid \hat{Z} = \hat{z})$, and introduce the matrix L defined by

$$L(z, \hat{z}) := \Phi(z, \hat{z})\, \mathbb{E}_{\hat{z}}\; m(\hat{z}, \hat{\varepsilon}). \tag{A1}$$

Here L is expressed as a function on $\mathsf{Z} \times \mathsf{Z}$ but can be represented in traditional matrix notation by enumerating Z. Specifically, if $\mathsf{Z} = \{z_1, \ldots, z_N\}$, then $L = \Phi D$, where $D := \operatorname{diag}\{\mathbb{E}_{z_1} M, \ldots, \mathbb{E}_{z_N} M\}$.

For a square matrix A, let $s(A)$ denote its spectral radius. In other words, $s(A) := \max_{\alpha \in \Lambda} |\alpha|$, where Λ is the set of eigenvalues of A.

Lemma A.1. *Given L defined in* (A1)*, the asymptotic yield satisfies $\kappa(M) = -\ln s(L)$.*

Proof. By induction, we can show that, for any function $h : \mathsf{Z} \to \mathbb{R}$ and $n \in \mathbb{N}$,

$$L^n h(z) = \mathbb{E}_z \left(\prod_{t=1}^{n} M_t \right) h(Z_n), \tag{A2}$$

where L^n is the n-th composition of the operator L with itself or, equivalently, the n-th power of the matrix L. By Theorem 9.1 of Krasnosel'skii et al. (2012) and the positivity of L, we have

$$s(L) = \lim_{n \to \infty} \|L^n h\|^{1/n}, \tag{A3}$$

where h is any function on Z that takes positive values, and $\|\cdot\|$ is any norm on the set of real-valued functions defined on Z. Letting $h \equiv 1$ and $\|f\| := \mathbb{E}\,|f(Z_0)|$, we obtain

$$s(L) = \lim_{n \to \infty} \left(\mathbb{E} \left| \mathbb{E}_{Z_0} \prod_{t=1}^{n} M_t \right| \right)^{1/n} = \lim_{n \to \infty} \left(\mathbb{E} \prod_{t=1}^{n} M_t \right)^{1/n} = \lim_{n \to \infty} q_n^{1/n},$$

where the first and the last equalities are by definition, and the second equality is due to the Markov property. Since the log function is continuous, we then have $\ln s(L) = \lim_{n \to \infty} \ln q_n / n = -\kappa(M)$ and the claim follows. $\square$

Corollary A.1. *Assumption 2.1 holds if and only if $s(L) < \mathrm{e}^{\delta}$.*

This follows directly from $\kappa(M) = -\ln s(L)$. Below we routinely use the alternative version $s(L) < \mathrm{e}^{\delta}$ for Assumption 2.1.

In the main text we imposed $p(b) < \infty$ to simplify analysis. Here and below, we relax this assumption. We assume instead $p(b) \leq \infty$, and prove that all the theoretical results in Sections 2–3 still hold in this generalized setup. To that end, we assume that

$$\mathbb{E}_z \max\{p(Y), 0\} < \infty \quad \text{for all } z \in \mathsf{Z}. \tag{A4}$$

(This mild assumption confines the expected market value of commodity output to be finite and holds trivially in the setting of Section 2, where p is bounded above.) We then update the endogenous state space X and define it as

$$\mathsf{X} := \begin{cases} (b, \infty), & \text{if } p(b) = \infty, \\ [b, \infty), & \text{if } p(b) < \infty. \end{cases}$$

There is no loss of generality to truncate the endogenous state space when $p(b) = \infty$, because in this case, (A4) implies that $Y_t > b$ almost surely, and thus $X_t > b$ with probability one for all t.

Let $p_0(x) := \max\{p(x), 0\}$ and let $\mathscr{C}$ be all continuous $f : \mathsf{S} \to \mathbb{R}$ such that f is decreasing in its first argument, $f(x, z) \geq p_0(x)$ for all $(x, z) \in \mathsf{S}$, and

$$\sup_{(x,z)\in\mathsf{S}} |f(x, z) - p_0(x)| < \infty.$$

Obviously, $\mathscr{C}$ reduces to the candidate space in Theorem 2.1 when the demand function p is bounded above, i.e., when $p(b) < \infty$. To compare pricing policies, we metrize $\mathscr{C}$ via

$$\rho(f, g) := \|f - g\| := \sup_{(x,z)\in\mathsf{S}} |f(x, z) - g(x, z)|.$$

Although f and g are not required to be bounded, one can show that ρ is a valid metric on $\mathscr{C}$ and that $(\mathscr{C}, \rho)$ is a complete metric space (see, e.g., Ma et al., 2020).

We aim to characterize the equilibrium pricing rule as the unique fixed point of the *equilibrium price operator* described as follows: For fixed $f \in \mathscr{C}$ and $(x, z) \in \mathsf{S}$, the value of Tf at (x, z) is defined as the $\xi \geq p_0(x)$ that solves

$$\xi = \psi(\xi, x, z) := \min\left\{\max\left\{\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f\left(\mathrm{e}^{-\delta} I(\xi, x, z) + \hat{Y}, \hat{Z}\right) - k,\ p(x)\right\},\ p(b)\right\}, \tag{A5}$$

where, considering free disposal,

$$I(\xi, x, z) := \begin{cases} x - p^{-1}(\xi), & \text{if } x < x_f^*(z) \\ x_f^*(z) - p^{-1}(\xi), & \text{if } x \geq x_f^*(z) \end{cases} \tag{A6}$$

with

$$x_f^*(z) := \inf\left\{x \geq p^{-1}(0) \colon \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f\left(\mathrm{e}^{-\delta}[x - p^{-1}(0)] + \hat{Y}, \hat{Z}\right) - k = 0\right\}.$$

The domain of ψ is

$$\mathsf{G} := \{(\xi, x, z) \in \mathbb{R}_+ \times \mathsf{S} \colon \xi \in B(x)\}, \tag{A7}$$

where $B(x)$ is defined for each x as

$$B(x) := \begin{cases} [p_0(x), \infty), & \text{if } p(b) = \infty, \\ [p_0(x), p(b)], & \text{if } p(b) < \infty. \end{cases} \tag{A8}$$

Proposition A.1. *If $f \in \mathscr{C}$ and $(x, z) \in \mathsf{S}$, then $Tf(x, z)$ is uniquely defined.*

Proof. Fix $f \in \mathscr{C}$ and $(x, z) \in \mathsf{S}$. Since f is decreasing in its first argument and p^{-1} is decreasing (by the inverse function theorem), the map $\xi \mapsto \psi(\xi, x, z)$ is decreasing. Since the left-hand-side of equation (A5) is strictly increasing in ξ, (A5) can have at most one solution. Hence, uniqueness holds. Existence follows from the intermediate value theorem provided we can show that

(a) $\xi \mapsto \psi(\xi, x, z)$ is a continuous function,
(b) there exists $\xi \in B(x)$ such that $\xi \leq \psi(\xi, x, z)$, and
(c) there exists $\xi \in B(x)$ such that $\xi \geq \psi(\xi, x, z)$.

For part (a), it suffices to show that

$$g(\xi) := \mathbb{E}_z \hat{M} f\left(\hat{Y} + \mathrm{e}^{-\delta} I(\xi, x, z), \hat{Z}\right)$$

is continuous on $B(x)$. To see this, fix $\xi \in B(x)$ and $\xi_n \to \xi$. Since $f \in \mathscr{C}$, there exists $D \in \mathbb{R}_+$ such that

$$0 \leq \hat{M} f\left(\hat{Y} + \mathrm{e}^{-\delta} I(\xi_n, x, z), \hat{Z}\right) \leq \hat{M} f(\hat{Y}, \hat{Z}) \leq \hat{M}\,[p_0(\hat{Y}) + D].$$

Since $\mathbb{E}_z \hat{M} p_0(\hat{Y}) = \mathbb{E}_z \left[\mathbb{E}_{\hat{z}}\, \hat{M}\ \mathbb{E}_{\hat{z}}\, p_0(\hat{Y})\right]$, the last term is integrable by (A4). Hence, the dominated convergence theorem applies. From this fact and the continuity of f, p^{-1}, and I, we obtain $g(\xi_n) \to g(\xi)$. Hence, $\xi \mapsto \psi(\xi, x, z)$ is continuous.

Regarding part (b), consider $\xi = p_0(x)$. If $p(x) \geq 0$, then $\xi = p(x)$ and thus

$$\psi(\xi, x, z) \geq \min\{p(x), p(b)\} = p(x) = \xi.$$

If $p(x) < 0$, then $\xi = 0$. In this case, $I(\xi, x, z) = I(0, x, z) \leq x_f^*(z) - p^{-1}(0)$. The monotonicity of f and the definition of x_f^* then imply that

$$\mathrm{e}^{-\delta}\, \mathbb{E}_z \hat{M} f\left(\mathrm{e}^{-\delta} I(\xi, x, z) + \hat{Y}, \hat{Z}\right) - k \geq \mathrm{e}^{-\delta}\, \mathbb{E}_z \hat{M} f\left(\mathrm{e}^{-\delta}[x_f^*(z) - p^{-1}(0)] + \hat{Y}, \hat{Z}\right) - k = 0.$$

By the definition of ψ,

$$\psi(\xi, x, z) \geq \min\{\max\{0, p(x)\}, p(b)\} = \min\{0, p(b)\} = 0 = \xi.$$

We have now verified part (b).

If $p(b) < \infty$, then part (c) holds by letting $\xi = p(b)$. If $p(b) = \infty$, then part (c) holds as ξ gets large since $\xi \mapsto \psi(\xi, x, z)$ is decreasing and bounded.

In summary, we have verified both existence and uniqueness. □

Proposition A.2. *$Tf \in \mathscr{C}$ for all $f \in \mathscr{C}$.*

Proof. Fix $f \in \mathscr{C}$ and define $g(\xi, x, z) := \mathbb{E}_z \hat{M} f \left(\hat{Y} + \mathrm{e}^{-\delta} I(\xi, x, z), \hat{Z}\right)$.

First, we show that Tf is continuous. To this end, we first show that ψ in (A5) is jointly continuous on the set G defined in (A7). This will be true if g is jointly continuous on G. For any $\{(\xi_n, x_n, z_n)\}$ and (ξ, x, z) in G with $(\xi_n, x_n, z_n) \to (\xi, x, z)$, we need to show that $g(\xi_n, x_n, z_n) \to g(\xi, x, z)$. Define

$$h_1(\xi, x, z, \hat{Z}, \hat{\varepsilon}, \hat{\eta}),\ h_2(\xi, x, z, \hat{Z}, \hat{\varepsilon}, \hat{\eta}) := \hat{M} f(\hat{Y}, \hat{Z}) \pm \hat{M} f \left(\hat{Y} + \mathrm{e}^{-\delta} I(\xi, x, z), \hat{Z}\right),$$

where $\hat{M} := m(\hat{Z}, \hat{\varepsilon})$ and $\hat{Y} = y(\hat{Z}, \hat{\eta})$. Then h_1 and h_2 are continuous in $(\xi, x, z, \hat{Z})$ by the continuity of f, p^{-1}, and I, and non-negative by the monotonicity of f in its first argument.

Let π_ε and π_η denote respectively the probability measure of $\{\varepsilon_t\}$ and $\{\eta_t\}$. Fatou's lemma and Theorem 1.1 of Feinberg et al. (2014) imply that

$$\begin{aligned}
&\int \int \sum_{\hat{z} \in \mathsf{Z}} h_i(\xi, x, z, \hat{z}, \hat{\varepsilon}, \hat{\eta}) \Phi(z, \hat{z}) \pi_\varepsilon \,\mathrm{d}(\hat{\varepsilon}) \pi_\eta \,\mathrm{d}(\hat{\eta}) \\
&\leq \int \int \liminf_{n \to \infty} \sum_{\hat{z} \in \mathsf{Z}} h_i(\xi_n, x_n, z_n, \hat{z}, \hat{\varepsilon}, \hat{\eta}) \Phi(z_n, \hat{z}) \pi_\varepsilon(\mathrm{d}\,\hat{\varepsilon}) \pi_\eta(\mathrm{d}\,\hat{\eta}) \\
&\leq \liminf_{n \to \infty} \int \int \sum_{\hat{z} \in \mathsf{Z}} h_i(\xi_n, x_n, z_n, \hat{z}, \hat{\varepsilon}, \hat{\eta}) \Phi(z_n, \hat{z}) \pi_\varepsilon(\mathrm{d}\,\hat{\varepsilon}) \pi_\eta(\mathrm{d}\,\hat{\eta}).
\end{aligned}$$

Since in addition $z \mapsto \mathbb{E}_z \hat{M} f(\hat{Y}, \hat{Z})$ is continuous, we have

$$\pm \mathbb{E}_z \hat{M} f \left(\hat{Y} + \mathrm{e}^{-\delta} I(\xi, x, z),\ \hat{Z}\right) \leq \liminf_{n \to \infty} \left(\pm \mathbb{E}_{z_n} \hat{M} f \left(\hat{Y} + \mathrm{e}^{-\delta} I(\xi_n, x_n, z_n),\ \hat{Z}\right)\right).$$

Then g is continuous, since the above inequality is equivalent to

$$\limsup_{n \to \infty} g(\xi_n, x_n, z_n) \leq g(\xi, x, z) \leq \liminf_{n \to \infty} g(\xi_n, x_n, z_n).$$

Hence, ψ is continuous on G, as was to be shown. Since $\xi \mapsto \psi(\xi, x, z)$ takes values in

$$\Gamma(x, z) := \left[p_0(x),\ \min\left\{p(b),\ p_0(x) + \mathrm{e}^{-\delta}\, \mathbb{E}_z \hat{M}(p_0(\hat{Y}) + D)\right\}\right]$$

for some $D \in \mathbb{R}_+$, and the correspondence $(x, z) \mapsto \Gamma(x, z)$ is nonempty, compact-valued and continuous, Theorem B.1.4 of Stachurski (2009) implies that Tf is continuous on S.

Second, we show that Tf is decreasing in x. Suppose for some $z \in \mathsf{Z}$ and x_1, $x_2 \in \mathsf{X}$ with $x_1 < x_2$, we have $\xi_1 := Tf(x_1, z) < Tf(x_2, z) =: \xi_2$. Since f is decreasing in its first argument by assumption and I defined in (A6) is increasing in ξ and x, ψ is decreasing in ξ and x. Then $\xi_2 > \xi_1 = \psi(\xi_1, x_1, z) \geq \psi(\xi_2, x_2, z) = \xi_2$, which is a contradiction.

Third, we show that $\sup_{(x,z)\in \mathsf{S}} |Tf(x, z) - p_0(x)| < \infty$. This obviously holds since

$$
\begin{aligned}
&|Tf(x,z) - p_0(x)| = Tf(x,z) - p_0(x) \\
&\leq \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f\left(\hat{Y} + \mathrm{e}^{-\delta} I\left(Tf(x,z), x, z\right), \hat{Z}\right) \leq \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}[p_0(\hat{Y}) + D]
\end{aligned}
$$

for all $(x, z) \in \mathsf{S}$ and some $D \in \mathbb{R}_+$, and the last term is finite by (A4).

Finally, Proposition A.1 implies that $Tf(x, z) \in B(x)$ for all $(x, z) \in \mathsf{S}$. In conclusion, we have shown that $Tf(x, z) \in \mathscr{C}$. □

Lemma A.2. *T is order preserving on $\mathscr{C}$. That is, $Tf_1 \leq Tf_2$ for all $f_1, f_2 \in \mathscr{C}$ with $f_1 \leq f_2$.*

Proof. Let f_1, f_2 be functions in $\mathscr{C}$ with $f_1 \leq f_2$. Recall ψ defined in (A5). With a slight abuse of notation, we define ψ_f such that $\psi_f(Tf(x, z), x, z) = Tf(x, z)$ for $f \in \{f_1, f_2\}$. Then $f_1 \leq f_2$ implies that $\psi_{f_1} \leq \psi_{f_2}$. Suppose to the contrary that there exits $(x, z) \in \mathsf{S}$ such that $\xi_1 := Tf_1(x, z) > Tf_2(x, z) = \xi_2$.

Since we have shown that $\xi \mapsto \psi(\xi, x, z)$ is decreasing for each $f \in \mathscr{C}$ and $(x, z) \in \mathsf{S}$, we have $\xi_1 = \psi_{f_1}(\xi_1, x, z) \leq \psi_{f_2}(\xi_1, x, z) \leq \psi_{f_2}(\xi_2, x, z) = \xi_2$, which is a contradiction. Therefore, T is order preserving. □

Lemma A.3. *There exist $N \in \mathbb{N}$ and $\alpha < 1$ such that, for all $n \geq N$,*

$$
\max_{z\in \mathsf{Z}} \mathbb{E}_z \prod_{t=1}^{n} \mathrm{e}^{-\delta} M_t < \alpha^n.
$$

Moreover, $D_1 := \sum_{t=0}^{\infty} \max_{z\in \mathsf{Z}} \mathbb{E}_z \prod_{i=1}^{t} \mathrm{e}^{-\delta} M_i < \infty$.

Proof. The second inequality follows immediately from the first inequality. To verify the first inequality, note that letting $h \equiv 1$ and $\|f\| = \max_{z\in \mathsf{Z}} |f(z)|$ in (A3) yields

$$
s(L) = \lim_{n\to\infty} \left(\max_{z\in \mathsf{Z}} \mathbb{E}_z \prod_{t=1}^{n} M_t \right)^{1/n}.
$$

Since $\mathrm{e}^{-\delta} s(L) < 1$ by Corollary A.1, there exists $N \in \mathbb{N}$ and $\alpha < 1$ such that for all $n \geq N$,

$$
\mathrm{e}^{-\delta} \left(\max_{z\in \mathsf{Z}} \mathbb{E}_z \prod_{t=1}^{n} M_t \right)^{1/n} = \left(\max_{z\in \mathsf{Z}} \mathbb{E}_z \prod_{t=1}^{n} \mathrm{e}^{-\delta} M_t \right)^{1/n} < \alpha.
$$

Hence, the first inequality holds, and the proof is now complete. □

To simplify notation, for given $\hat{Y}$, we denote

$$h(\xi, x, z) := \hat{Y} + \mathrm{e}^{-\delta} I(\xi, x, z) \quad \text{and} \quad g(\zeta, x) := \min\{\max\{\zeta, p(x)\}, p(b)\}.$$

By definition, $\xi \mapsto h(\xi, x, z)$ and $\zeta \mapsto g(\zeta, x)$ are increasing given (x, z).

Lemma A.4. *For all $m \in \mathbb{N}$ and $(x, z) \in \mathsf{S}$, we have*

$$T^m(f+\gamma)(x, z) \leq T^m f(x, z) + \gamma \, \mathbb{E}_z \prod_{t=1}^{m} \mathrm{e}^{-\delta} M_t. \tag{A9}$$

Proof. Fix $f \in \mathscr{C}$, $\gamma \geq 0$, and let $f_\gamma(x, z) := f(x, z) + \gamma$. By the definition of T,

$$\begin{aligned}
T f_\gamma(x, z) &= g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f_\gamma\left(h[T f_\gamma(x, z), x, z], \hat{Z}\right), x\right] \\
&\leq g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f\left(h[T f_\gamma(x, z), x, z], \hat{Z}\right), x\right] + \gamma\, \mathbb{E}_z\, \mathrm{e}^{-\delta} \hat{M} \\
&\leq g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f\left(h[T f(x, z), x, z], \hat{Z}\right), x\right] + \gamma\, \mathbb{E}_z\, \mathrm{e}^{-\delta} \hat{M},
\end{aligned}$$

where the second inequality is due to the fact that $f \leq f_\gamma$ and T is order preserving. Hence, $T(f+\gamma)(x, z) \leq T f(x, z) + \gamma\, \mathbb{E}_z\, \mathrm{e}^{-\delta} \hat{M}$ and (A9) holds for $m = 1$. Suppose (A9) holds for arbitrary m. It remains to show that it holds for $m+1$. For $z \in \mathsf{Z}$, let $\ell(z) := \gamma\, \mathbb{E}_z \prod_{t=1}^{m} \mathrm{e}^{-\delta} M_t$. By the induction hypothesis, Lemma A.2, and the Markov property,

$$\begin{aligned}
T^{m+1} f_\gamma(x, z) &= g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}(T^m f_\gamma)\left(h[T^{m+1} f_\gamma(x, z), x, z], \hat{Z}\right), x\right] \\
&\leq g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}(T^m f + \ell)\left(h[T^{m+1} f_\gamma(x, z), x, z], \hat{Z}\right), x\right] \\
&\leq g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}(T^m f)\left(h[T^{m+1} f_\gamma(x, z), x, z], \hat{Z}\right), x\right] + \mathbb{E}_z\, \mathrm{e}^{-\delta} M_1 \ell(Z_1) \\
&\leq T^{m+1} f(x, z) + \gamma\, \mathbb{E}_z\, \mathrm{e}^{-\delta} M_1\, \mathbb{E}_{Z_1}\, \mathrm{e}^{-\delta} M_1 \cdots \mathrm{e}^{-\delta} M_m \\
&= T^{m+1} f(x, z) + \gamma\, \mathbb{E}_z \prod_{t=1}^{m+1} \mathrm{e}^{-\delta} M_t.
\end{aligned}$$

Hence (A9) holds by induction. □

Theorem A.1. *If Assumption 2.1 holds, then T is well defined on the function space $\mathscr{C}$, and there exists an $n \in \mathbb{N}$ such that T^n is a contraction mapping on $(\mathscr{C}, \rho)$. Moreover,*

(i) *T has a unique fixed point f^* in $\mathscr{C}$.*

(ii) *The fixed point f^* is the unique equilibrium pricing rule in $\mathscr{C}$.*

(iii) *For each f in $\mathscr{C}$, we have $\rho(T^k f, f^*)$ as $k \to \infty$.*

Proof. Proposition A.1 shows that T is a well-defined operator on $\mathscr{C}$. Since T is order preserving by Lemma A.2 and $\mathscr{C}$ is closed under the addition of non-negative constants, to show that T^n is a contraction mapping on $(\mathscr{C}, \rho)$ for some $n \in \mathbb{N}$, based on Blackwell (1965), it remains to verify the existence of $n \in \mathbb{N}$ and $\theta < 1$ such that $T^n(f+\gamma) \leq$

$T^n f + \theta\gamma$ for all $f \in \mathscr{C}$ and $\gamma \geq 0$. This obviously holds by Lemmas A.3–A.4, implying that T^n is a contraction on $(\mathscr{C}, \rho)$ of modulus θ. Claims (i)–(iii) then follow from the Banach contraction mapping theorem and the definition of the equilibrium pricing rule. □

For each f in $\mathscr{C}$, we define

$$\bar{p}_f^0(z) := \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f(\hat{Y}, \hat{Z}) - k \quad \text{and} \quad \bar{p}_f(z) := \min\{\bar{p}_f^0(z), p(b)\}.$$

Lemma A.5. *For each f in $\mathscr{C}$, Tf satisfies*

(i) $Tf(x, z) = p(x)$ *if and only if* $x \leq p^{-1}[\bar{p}_f(z)]$,

(ii) $Tf(x, z) > p_0(x)$ *if and only if* $p^{-1}[\bar{p}_f(z)] < x < x_f^*(z)$, *and*

(iii) $Tf(x, z) = 0$ *if and only if* $x \geq x_f^*(z)$.

Proof. Regarding claim (i), suppose $Tf(x, z) = p(x)$. We show that $x \leq p^{-1}[\bar{p}_f(z)]$. Note that in this case, $x \leq p^{-1}(0) \leq x_f^*(z)$ since $p(x) = Tf(x, z) \geq 0$. Hence,

$$I\,[Tf(x, z), x, z] = x - p^{-1}[Tf(x, z)] = 0.$$

When $x = b$, the proof is trivial. When $x > b$, we have $Tf(x, z) = p(x) < p(b)$. By the definition of T,

$$\begin{aligned} p(x) = Tf(x, z) &= \max\left\{\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f\left(\mathrm{e}^{-\delta} I\,[Tf(x, z), x, z] + \hat{Y}, \hat{Z}\right) - k, p(x)\right\} \\ &= \max\left\{\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f(\hat{Y}, \hat{Z}) - k, p(x)\right\} = \max\left\{\bar{p}_f^0(z), p(x)\right\}. \end{aligned}$$

This implies that $p(x) \geq \bar{p}_f^0(z) \geq \bar{p}_f(z)$. Hence $x \leq p^{-1}[\bar{p}_f(z)]$.

Next, we prove that $x \leq p^{-1}[\bar{p}_f(z)]$ implies $Tf(x, z) = p(x)$. If $\bar{p}_f^0(z) \geq p(b)$, then

$$\bar{p}_f(z) = p(b) \quad \Longrightarrow \quad x \leq p^{-1}[\bar{p}_f(z)] = p^{-1}[p(b)] = b.$$

Hence $x = b$. Then by definition $Tf(x, z) = \min\{\bar{p}_f^0(z), p(b)\} = p(b) = p(x)$.

If $\bar{p}_f^0(z) < p(b)$, then $\bar{p}_f(z) = \bar{p}_f^0(z)$. Since in addition

$$\bar{p}_f^0(z) \geq \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} p(\hat{Y}) - k > 0 \quad \text{and} \quad x \leq p^{-1}[\bar{p}_f(z)],$$

we have $x < p^{-1}(0) \leq x_f^*(z)$ in this case. Suppose to the contrary that $Tf(x, z) > p(x)$ for some $(x, z) \in \mathsf{S}$. Then by the definition of T,

$$p(x) < \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f\left[\mathrm{e}^{-\delta}\left(x - p^{-1}[Tf(x, z)]\right) + \hat{Y}, \hat{Z}\right] - k.$$

The monotonicity of f in its first argument then implies that

$$p(x) < \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f(\hat{Y}, \hat{Z}) - k = \bar{p}_f^0(z) = \bar{p}_f(z),$$

which is a contradiction. Claim (i) is now verified.

Note that claim (ii) follows immediately once claim (iii) is verified. To see that claim (iii) is true, suppose to the contrary that $x \geq x_f^*(x)$ and $Tf(x,z) > 0$ for some $(x,z) \in \mathsf{S}$. Then

$$I\left[Tf(x,z),x,z\right] = x_f^*(z) - p^{-1}[Tf(x,z)] > x_f^*(z) - p^{-1}(0).$$

By the definition of $x_f^*(z)$ and the monotonicity of f, this gives

$$\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}f\left(\mathrm{e}^{-\delta}I\left[Tf(x,z),x,z\right]+\hat{Y},\hat{Z}\right) - k \leq 0.$$

Using the definition of T, we obtain $0 < Tf(x,z) \leq \min\{\max\{0,p(x)\},p(b)\} = 0$, which is a contradiction. Hence, $x \geq x_f^*(z)$ implies $Tf(x,z) = 0$.

Now suppose $Tf(x,z) = 0$. The definition of T implies that

$$\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}f\left(\mathrm{e}^{-\delta}I(0,x,z)+\hat{Y},\hat{Z}\right) - k \leq 0.$$

By the definition of $x_f^*(z)$, this gives $x \geq x_f^*(z)$. Claim (iii) is now verified. □

Proof of Theorem 2.1. Theorem A.1 implies that there exists a unique equilibrium pricing rule f^* in $\mathscr{C}$. Claims (i)–(iii) follow immediately from Lemma A.5 since $\bar{p}(z) = \bar{p}_{f^*}(z)$ and f^* is the unique fixed point of T in $\mathscr{C}$.

To see that claim (iv) holds, suppose $f^*(x,z)$ is not strictly decreasing under the given conditions. Then by claims (i)–(iii), there exists $z \in \mathsf{Z}$ with $\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M} < 1$ and a first interval $[x_0,x_1] \subset (p^{-1}[\bar{p}(z)], x^*(z))$ such that $f^*(x,z) \equiv B$ on this interval for some constant $B > 0$. By the definition of T, for all $x \in [x_0,x_1]$,

$$B = f^*(x,z) = \mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}f\left(\mathrm{e}^{-\delta}I\left[f^*(x,z),x,z\right]+\hat{Y},\hat{Z}\right) - k.$$

Since the left-hand-side is a constant, $f\left(\mathrm{e}^{-\delta}I\left[f^*(x,z),x,z\right]+\hat{Y},\hat{Z}\right) = B'$ for some constant B'. Moreover, $B' \leq B$ since f is decreasing in x and $[x_0,x_1]$ is the first interval on which f is constant. Since in addition $\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M} < 1$, we have $B \leq \mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}B - k < B - k \leq B$, which is a contradiction. Hence claim (iv) must be true. □

Proof of Proposition 2.1. The continuity of i^* and claims (i)–(iii) follow from Theorem 2.1 and the definition of i^*. We next show that $i^*(x,z)$ is increasing in x. Since $i^*(x,z)$ is constant given z when $x \leq p^{-1}[\bar{p}(z)]$ and when $x \geq x^*(z)$ by claim (i) and claim (iii), it remains to show that $i^*(x,z)$ is increasing in x when $p^{-1}[\bar{p}(z)] < x < x^*(z)$. In this case $i^*(x,z) = x - p^{-1}[f^*(x,z)]$. Suppose to the contrary that there exist $z \in \mathsf{Z}$ and $x_1, x_2 \in (p^{-1}[\bar{p}(z)], x^*(z))$ such that $x_1 < x_2$ and $i^*(x_1,z) > i^*(x_2,z)$. Then by definition,

$$x_1 - p^{-1}[f^*(x_1,z)] > x_2 - p^{-1}[f^*(x_2,z)].$$

Since $x_1 < x_2$, this gives $p^{-1}[f^*(x_2, z)] > p^{-1}[f^*(x_1, z)]$. But by (ii) of Theorem 2.1 and the definition of T, we obtain

$$\begin{aligned} f^*(x_1, z) &= \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f^*\left(\mathrm{e}^{-\delta} i^*(x_1, z) + \hat{Y}, \hat{Z}\right) - k \\ &\leq \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M} f^*\left(\mathrm{e}^{-\delta} i^*(x_2, z) + \hat{Y}, \hat{Z}\right) - k \leq f^*(x_2, z), \end{aligned}$$

which implies $p^{-1}[f^*(x_1, z)] \geq p^{-1}[f^*(x_2, z)]$. This is a contradiction. Hence, it is true that $i^*(x, z)$ is increasing in x.

It remains to verify claim (iv). Pick any $z \in \mathsf{Z}$ and $x_1, x_2 \in (p^{-1}[\bar{p}(z)], x^*(z))$ with $x_1 < x_2$. By claim (iv) of Theorem 2.1, we have $f^*(x_1, z) > f^*(x_2, z)$. Using the definition of T and claim (ii) of Theorem 2.1 again, we have

$$\mathbb{E}_z\, \hat{M} f^*\left(\mathrm{e}^{-\delta} i^*(x_1, z) + \hat{Y}, \hat{Z}\right) > \mathbb{E}_z\, \hat{M} f^*\left(\mathrm{e}^{-\delta} i^*(x_2, z) + \hat{Y}, \hat{Z}\right).$$

The monotonicity of f^* then gives $i^*(x_1, z) < i^*(x_2, z)$. Hence claim (iv) holds. □

B. Proof of Section 3 Results

A Markov chain $\{Z_t\}$ with transition matrix F is called *monotone* if

$$\int h(\hat{z})\, \mathrm{d}\, F(z_1, \hat{z}) \leq \int h(\hat{z})\, \mathrm{d}\, F(z_2, \hat{z})$$

whenever $z_1 \leq z_2$ and $h : \mathsf{Z} \to \mathbb{R}$ is bounded and increasing.

Proof of Proposition 3.1. Let T_1 and T_2 be respectively the equilibrium price operators corresponding to $\{R_{1t}\}$ and $\{R_{2t}\}$. It suffices to show that $T_1 f \leq T_2 f$ for all $f \in \mathscr{C}$. To see this, we adopt an induction argument. Suppose $T_1^k f \leq T_2^k f$. Then by the order preserving property of the equilibrium price operator and the initial argument $T_1 f \leq T_2 f$ for all f in $\mathscr{C}$, we have $T_1^{k+1} f = T_1(T_1^k f) \leq T_1(T_2^k f) \leq T_2(T_2^k f) = T_2^{k+1} f$. Hence, $T_1^k f \leq T_2^k f$ for all $k \in \mathbb{N}$ and $f \in \mathscr{C}$. Letting $k \to \infty$ then yields $f_1^* \leq f_2^*$.

We now show that $T_1 f \leq T_2 f$ for all $f \in \mathscr{C}$. Suppose there exists $(x, z) \in \mathsf{S}$ such that $\xi_1 := T_1 f(x, z) > T_2 f(x, z) =: \xi_2$. Let $M_{it} = 1/R_{it}$ for $i = 1, 2$. Since $R_{1t} \geq R_{2t}$, $M_{1t} \leq M_{2t}$. The monotonicity of g and h (recall (A)) then imply that

$$\begin{aligned} \xi_1 &= g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}_1 f\left(h(\xi_1, x), \hat{Z}\right), x\right] \\ &\leq g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}_2 f\left(h(\xi_1, x), \hat{Z}\right), x\right] \leq g\left[\mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}_2 f\left(h(\xi_2, x), \hat{Z}\right), x\right] = \xi_2, \end{aligned}$$

which is a contradiction. Therefore, $T_1 f \leq T_2 f$ and all the stated claims hold. □

Proof of Proposition 3.2. Let $\mathscr{C}_1$ be the elements in $\mathscr{C}$ such that $z \mapsto f(x, z)$ is decreasing for all x. Obviously, $\mathscr{C}_1$ is a closed subset of $\mathscr{C}$. Therefore, to show that $z \mapsto f^*(x, z)$ is decreasing for all x, it suffices to verify $T\mathscr{C}_1 \subset \mathscr{C}_1$.

Fix $f \in \mathscr{C}_1$ and $z_1, z_2 \in \mathsf{Z}$ with $z_1 \leq z_2$. Suppose there exists an x such that

$$\xi_1 := Tf(x, z_1) < Tf(x, z_2) =: \xi_2. \tag{A10}$$

Note that f is a decreasing function since $f \in \mathscr{C}_1$. Moreover, by assumption $m(z, \varepsilon) = 1/r(z, \varepsilon)$ is decreasing in z, $y(z, \eta)$ is increasing in z, and Φ is monotone. Therefore, for all $\xi \in B(x)$, we have

$$\mathbb{E}_{z_1} \hat{M} f \left(\mathrm{e}^{-\delta}[x - p^{-1}(\xi)] + \hat{Y}, \hat{Z} \right) \geq \mathbb{E}_{z_2} \hat{M} f \left(\mathrm{e}^{-\delta}[x - p^{-1}(\xi)] + \hat{Y}, \hat{Z} \right). \tag{A11}$$

In particular, by the definition of x_f^*, we have $x_f^*(z_2) \leq x_f^*(z_1)$. If $x < x_f^*(z_2)$, then

$$I(\xi_1, x, z_1) = x - p^{-1}(\xi_1) \leq x - p^{-1}(\xi_2) = I(\xi_2, x, z_2).$$

Recall ψ defined in (A5). The above inequality and (A11) imply that

$$\xi_1 = \psi(\xi_1, x, z_1) \geq \psi(\xi_1, x, z_2) \geq \psi(\xi_2, x, z_2) = \xi_2.$$

If $x \geq x_f^*(z_2)$, then we also have $\xi_1 \geq \xi_2$ since $\xi_2 = 0$ and $\xi_1 \geq 0$. In either case, this is contradicted with (A10). Therefore, we have shown that $z \mapsto Tf(x, z)$ is decreasing for all x and $T\mathscr{C}_1 \subset \mathscr{C}_1$. It then follows that $z \mapsto f^*(x, z)$ is decreasing for all x.

To see that $i^*(x, z)$ is decreasing in z, pick any $z_1, z_2 \in \mathsf{Z}$ with $z_1 \leq z_2$. By the definition of $x^*(z)$ and the monotonicity of $f^*(x, z)$ in z, we have

$$0 = f^*(x^*(z_1), z_1) \geq f^*(x^*(z_1), z_2)$$

and thus $x^*(z_1) \geq x^*(z_2)$. The definition of i^* and the monotonicity of p^{-1} and f^* then implies that

$$\begin{aligned} i^*(x, z_1) &= \min\{x, x^*(z_1)\} - p^{-1}[f^*(x, z_1)] \\ &\geq \min\{x, x^*(z_2)\} - p^{-1}[f^*(x, z_2)] = i^*(x, z_2). \end{aligned}$$

Hence $z \mapsto i^*(x, z)$ is decreasing for all x.

Finally, note that $\hat{Z} \mapsto \hat{M} f^*(\hat{Y}, \hat{Z}) = m(\hat{Z}, \hat{\varepsilon}) f^*(y(\hat{Z}, \hat{\eta}), \hat{Z})$ is decreasing because f^* is decreasing, y is increasing in z, and m is decreasing in z. Since in addition $\{Z_t\}$ is monotone, it follows immediately by definition that $z \mapsto \mathbb{E}_z \hat{M} f^*(\hat{Y}, \hat{Z})$ is decreasing. Hence $\bar{p}$ is decreasing by definition. □

Next, we discuss the correlation between commodity price and stochastic discount factor. To state the result, we suppose $Z_t = (Z_{1t}, \ldots, Z_{nt})$ takes values in $\mathbb{R}^n$. The following is a simple corollary of the key result of Fortuin et al. (1971).

Lemma B.1 (Fortuin–Kasteleyn–Ginibre)**.** *If f, g are decreasing integrable functions on $\mathbb{R}^n$ and $W = (W_1, \cdots, W_n)$ is a random vector on $\mathbb{R}^n$ such that $\{W_1, \cdots, W_n\}$ are independent, then $\mathbb{E}\, f(W)\, \mathbb{E}\, g(W) \leq \mathbb{E}\, f(W) g(W)$.*

Lemma B.1 implies that if f is decreasing and g is nondecreasing (so that $-g$ is decreasing), then we have $\mathbb{E} f(W) \mathbb{E} g(W) \geq \mathbb{E} f(W) g(W)$.

Proposition B.1. *If $m(z, \varepsilon)$ is decreasing in z, $y(z, \eta)$ is nondecreasing in z, Φ is monotone, and $\{Z_{1t}, \ldots, Z_{nt}\}$ are independent for each fixed t, then $\mathrm{Cov}_{t-1}(P_t, M_t) \geq 0$ for all $t \in \mathbb{N}$.*

Proof. The equilibrium path is $X_t = \mathrm{e}^{-\delta} i^*(X_{t-1}, Z_{t-1}) + y(Z_t, \eta_t)$ where

$$i^*(X_{t-1}, Z_{t-1}) = \min \{X_{t-1}, x^*(Z_{t-1})\} - p^{-1} [f^*(X_{t-1}, Z_{t-1})] .$$

Note that X_t is a nondecreasing function of Z_t since $z \mapsto y(z, \eta)$ is nondecreasing for all η. Moreover, the proof of Proposition 3.2 implies that f^* is a decreasing function under the assumptions of the current proposition. Hence, $Z_t \mapsto f^*(X_t, Z_t)$ is decreasing. Since in addition $z \mapsto m(z, \varepsilon)$ is decreasing for all ε and $\{Z_{1t}, \ldots, Z_{nt}\}$ are independent, applying Lemma B.1 (taking $W = Z_t$) yields

$$\begin{aligned}
&\mathbb{E} [f^*(X_t, Z_t) m(Z_t, \varepsilon_t) \mid X_{t-1}, Z_{t-1}, \varepsilon_t, \eta_t] \\
&\geq \mathbb{E} [f^*(X_t, Z_t) \mid X_{t-1}, Z_{t-1}, \varepsilon_t, \eta_t] \, \mathbb{E} [m(Z_t, \varepsilon_t) \mid X_{t-1}, Z_{t-1}, \varepsilon_t, \eta_t] \\
&= \mathbb{E} [f^*(X_t, Z_t) \mid X_{t-1}, Z_{t-1}, \eta_t] \, \mathbb{E} [m(Z_t, \varepsilon_t) \mid Z_{t-1}, \varepsilon_t] .
\end{aligned}$$

Using this result, it follows that

$$\begin{aligned}
&\mathbb{E} (P_t M_t \mid X_{t-1}, Z_{t-1}) = \mathbb{E} [f^*(X_t, Z_t) m(Z_t, \varepsilon_t) \mid X_{t-1}, Z_{t-1}] \\
&= \mathbb{E} (\mathbb{E} [f^*(X_t, Z_t) m(Z_t, \varepsilon_t) \mid X_{t-1}, Z_{t-1}, \varepsilon_t, \eta_t] \mid X_{t-1}, Z_{t-1}) \\
&\geq \mathbb{E} \{\mathbb{E} [f^*(X_t, Z_t) \mid X_{t-1}, Z_{t-1}, \eta_t] \, \mathbb{E} [m(Z_t, \varepsilon_t) \mid Z_{t-1}, \varepsilon_t] \mid X_{t-1}, Z_{t-1}\} \\
&= \mathbb{E} [f^*(X_t, Z_t) \mid X_{t-1}, Z_{t-1}] \, \mathbb{E} [m(Z_t, \varepsilon_t) \mid X_{t-1}, Z_{t-1}] \\
&= \mathbb{E} (P_t \mid X_{t-1}, Z_{t-1}) \, \mathbb{E} (M_t \mid X_{t-1}, Z_{t-1}) ,
\end{aligned}$$

where the second-to-last equality holds because η_t is independent of ε_t. Hence,

$$\begin{aligned}
&\mathrm{Cov}_{t-1}(P_t, M_t) = \mathrm{Cov}(P_t, M_t \mid X_{t-1}, Z_{t-1}) \\
&= \mathbb{E} (P_t, M_t \mid X_{t-1}, Z_{t-1}) - \mathbb{E} (P_t \mid X_{t-1}, Z_{t-1}) \, \mathbb{E} (M_t \mid X_{t-1}, Z_{t-1}) \geq 0,
\end{aligned}$$

as was to be shown. □

Proof of Proposition 3.3. Since $R_t = 1/M_t$, applying Lemma B.1 again and working through similar steps to the proof of Proposition B.1, we can show that $\mathrm{Cov}_{t-1}(P_t, R_t) \leq 0$ for all t. The details are omitted. □

C. Positive Correlation

Here we provide examples showing that Proposition 3.3 does not hold in general if Z_t is positively or negatively correlated across dimensions. We begin with the following.

Proposition C.1. *If Assumption 2.1 holds and the inverse demand function is $p(x) = a + dx$ with $a > 0$ and $d < 0$, then the equilibrium pricing rule $f^*(x, z)$ is convex in x.*

Proof. Let $\mathscr{C}_2$ be the elements in $\mathscr{C}$ such that $x \mapsto f(x, z)$ is convex for all $z \in \mathsf{Z}$. Then $\mathscr{C}_2$ is a closed subset of $\mathscr{C}$. Hence it suffices to show that $T\mathscr{C}_2 \subset \mathscr{C}_2$. Fix $f \in \mathscr{C}_2$, since $Tf \in \mathscr{C}$ by Proposition A.2, it remains to show that $Tf(x, z)$ is convex in x. Since $Tf(x, z)$ is decreasing in x and, by Lemma A.5, $Tf(x, z)$ is linear in x when $x \leq p^{-1}[\bar{p}(z)]$ or $x \geq x_f^*(z)$, it suffices to show that $Tf(x, z)$ is convex in x on $B_0(z) := (p^{-1}[\bar{p}(z)], x_f^*(z))$. In this case,

$$Tf(x, z) = \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}f\left(\mathrm{e}^{-\delta}\left(x - p^{-1}[Tf(x, z)]\right) + \hat{Y}, \hat{Z}\right) - k.$$

Suppose to the contrary that $Tf(x, z)$ is not convex, then there exist $z \in \mathsf{Z}$, $x_1, x_2 \in B_0(z)$, and $\alpha \in [0, 1]$ such that, letting $x_0 := \alpha x_1 + (1 - \alpha)x_2$,

$$\begin{aligned}
&\alpha Tf(x_1, z) + (1 - \alpha)Tf(x_2, z) < Tf(x_0, z) \\
&= \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}f\left(\mathrm{e}^{-\delta}\left(x - p^{-1}[Tf(x_0, z)]\right) + \hat{Y}, \hat{Z}\right) - k \\
&\leq \mathrm{e}^{-\delta}\, \mathbb{E}_z\, \hat{M}f\left(\mathrm{e}^{-\delta}\left(x - p^{-1}\left[\alpha Tf(x_1, z) + (1 - \alpha)Tf(x_2, z)\right]\right) + \hat{Y}, \hat{Z}\right) - k \\
&\leq \alpha Tf(x_1, z) + (1 - \alpha)Tf(x_2, z),
\end{aligned}$$

where the last inequality is by convexity of $f(x, z)$ in x and the linearity of $p(x)$. This is a contradiction. Hence $Tf(x, z)$ is convex in x on $B_0(z)$ and the stated claim holds. □

Suppose $R_t = 0.98$ with probability 0.5 and $R_t = 1.02$ with probability 0.5. If $R_t = 0.98$, then $Y_t = y_0$ with probability one, and if $R_t = 1.02$, then $Y_t = y_1$ with probability φ and $Y_t = y_2$ with probability $1 - \varphi$. This is a special case of our framework. In particular,

$$\begin{gathered}
\varepsilon_t = \eta_t = 0, \quad Z_t = (Z_{1t}, Z_{2t}) = (R_t, Y_t), \\
r(Z_t, \varepsilon_t) = r(R_t, Y_t, \varepsilon_t) = R_t \quad \text{and} \quad y(Z_t, \eta_t) = y(R_t, Y_t, \eta_t) = Y_t.
\end{gathered}$$

Note that $\{Z_t\}$ is iid. Hence, it is naturally monotone and the equilibrium pricing rule is not a function of Z_t. Since in addition $r(z, \varepsilon)$ and $y(z, \eta)$ are increasing in z, the assumptions of Proposition 3.2 hold. However, because Z_{1t} and Z_{2t} (i.e., R_t and Y_t) are correlated, the assumptions of Proposition 3.3 are violated.

Some simple algebra shows that $\mathbb{E}\, R_t = 1$,

$$\mathbb{E}\, Y_t = \frac{y_0}{2} + \frac{\varphi y_1}{2} + \frac{(1 - \varphi)y_2}{2} \quad \text{and} \quad \mathbb{E}\, R_t Y_t = \frac{0.98 y_0}{2} + \frac{1.02\varphi y_1}{2} + \frac{1.02(1 - \varphi)y_2}{2}.$$

Hence $\mathrm{Cov}_{t-1}(R_t, Y_t) = \mathbb{E}\, R_t Y_t - \mathbb{E}\, R_t\, \mathbb{E}\, Y_t = -0.01 y_0 + 0.01\varphi y_1 + 0.01(1 - \varphi)y_2$. Choose δ such that $\delta > \log \mathbb{E}(1/R_t) \approx 0.0004$. Then $\beta := \mathrm{e}^{-\delta}\, \mathbb{E}(1/R_t) < 1$ and the following result holds under the current setup.

Lemma C.1. *Either (i) $P_t = 0$ for all t or (ii) $I_t = 0$ in finite time with probability one. If the per-unit storage cost $k > 0$, then (ii) holds.*

Proof. Suppose (ii) does not hold, then $I_t > 0$ for all t with positive probability, and the equilibrium price path satisfies

$$P_0 \leq e^{-\delta t}\, \mathbb{E}\left(\prod_{i=0}^{t} \frac{1}{R_i}\right) P_t - \left(\sum_{i=0}^{t-1} e^{-\delta i}\right) k \quad \text{for all } t. \tag{A12}$$

Note that $\{P_t\}$ is bounded with probability one since $f^* \in \mathscr{C}$ implies that, for some $L_0 < \infty$, we have $P_t = f^*(X_t) \leq f^*(Y_t) \leq f^*(\underline{y}) \leq p_0(\underline{y}) + L_0 =: L_1 < \infty$ with probability one, where $\underline{y} := \min\{y_0, y_1, y_2\}$. If in addition (i) does not hold, we may assume $P_0 > 0$ without loss of generality. In this case, (A12) implies that, when t is sufficiently large, we have $0 < P_0 \leq \beta^t L_1 < P_0$ with positive probability, which is a contradiction. Hence either (i) or (ii) holds. If, on the other hand, $k > 0$ and (ii) does not hold, then for sufficiently large t, (A12) implies that $P_0 < P_0$ with positive probability for all $P_0 \geq 0$, which is also a contradiction. Hence (ii) holds and the second claim is also verified. □

Consider a linear inverse demand function p as in Proposition C.1. If $P_t = 0$ for all t, then $\mathrm{Cov}(R_t, P_t) = \mathrm{Cov}_{t-1}(R_t, P_t) = 0$ for all t. Otherwise, $I_{t-1} = 0$ for some finite t, in which case $X_t = Y_t$, $P_t = f^*(Y_t)$, and thus

$$\begin{aligned}\mathrm{Cov}_{t-1}(R_t, P_t) &= \mathbb{E}_{t-1} R_t P_t - \mathbb{E} R_t\, \mathbb{E}_{t-1} P_t = \mathbb{E} R_t f^*(Y_t) - \mathbb{E} R_t\, \mathbb{E} f^*(Y_t) \\ &= -0.01 f^*(y_0) + 0.01\varphi f^*(y_1) + 0.01(1-\varphi) f^*(y_2).\end{aligned}$$

If $y_1, y_2 < y_0$, then $\mathrm{Cov}_{t-1}(R_t, Y_t) < 0$ and $\mathrm{Cov}_{t-1}(R_t, P_t) > 0$ based on the monotonicity of f^*. If on the other hand y_0, y_1 and y_2 satisfy[25] $y_1 < p^{-1}(\bar{p}) < y_0 < y_2$, then since f^* is convex by Proposition C.1, and $f^*(x) > p(x)$ whenever $x > p^{-1}(\bar{p})$ by Theorem 2.1,

$$\frac{f^*(y_0) - f^*(y_2)}{f^*(y_1) - f^*(y_2)} < \frac{y_2 - y_0}{y_2 - y_1}.$$

Hence φ can be chosen such that $y_2 - y_0 > \varphi(y_2 - y_1)$ and $f^*(y_0) - f^*(y_2) < \varphi[f^*(y_1) - f^*(y_2)]$. In particular, the above inequalities respectively imply that

$$\mathrm{Cov}_{t-1}(R_t, Y_t) > 0 \quad \text{and} \quad \mathrm{Cov}_{t-1}(R_t, P_t) > 0.$$

D. An Identification Equivalence Result

Consider an economy E with linear inverse demand function $p(x) = a + dx$ where $a > 0$ and $d < 0$. Let $\{Y_t\}$ be a stationary Markov process with transition probability Ψ. Let b be the lower bound of the total available supply in this economy.

[25]Since $e^{-\delta}\, \mathbb{E}(1/R_t) < 1$, we have $\bar{p} = \min\{e^{-\delta}\, \mathbb{E} f^*(\hat{Y})/\hat{R} - k, p(b)\} < p(b)$. Thus $p^{-1}(\bar{p}) > b$ and this choice of y_0, y_1, y_2 is feasible.

Let $\tilde{E}$ be another economy where the output process satisfies $\tilde{Y}_t = \mu + \sigma Y_t$ with $\sigma > 0$ and the transition probability of $\{\tilde{Y}_t\}$ satisfies[26]

$$\tilde{\Psi}(y, \hat{Y}) = \Psi\left(\frac{y-\mu}{\sigma}, \frac{\hat{Y}-\mu}{\sigma}\right). \tag{A13}$$

Moreover, let the lower bound of the total available supply of economy $\tilde{E}$ be $\tilde{b} = \mu + \sigma b$ and the inverse demand function be

$$\tilde{p}(x) = \left(a - \frac{d\mu}{\sigma}\right) + \frac{d}{\sigma}x. \tag{A14}$$

The remaining assumptions are the same across economies E and $\tilde{E}$.

Proposition D.1. *$\tilde{E}$ and E generate the same commodity price process.*

Proof. To simplify notation, let f and i be the equilibrium pricing function and the equilibrium inventory function of the baseline economy E. Without loss of generality, we may assume $Z_t = Y_t$.[27] Then for all $(x, y) \in \mathsf{S}$, $\{f(x, y), i(x, y)\}$ is the unique solution to

$$f(x, y) = \min\left\{\max\left\{\mathrm{e}^{-\delta}\,\mathbb{E}_y\,\hat{M} f[\mathrm{e}^{-\delta} i(x, y) + \hat{Y}, \hat{Y}] - k,\ p(x)\right\}, p(b)\right\} \tag{A15}$$

$$i(x, y) = \begin{cases} x - p^{-1}[f(x, y)], & x < x_f(y) \\ x_f(y) - p^{-1}(0), & x \geq x_f(y) \end{cases} \tag{A16}$$

where

$$x_f(y) := \inf\left\{x \geq p^{-1}(0) : f(x, y) = 0\right\}.$$

Consider economy $\tilde{E}$, where all magnitudes are denoted with tildes. Let

$$\tilde{x} = \mu + \sigma x, \quad \tilde{y} = \mu + \sigma y,$$
$$\tilde{m}(\tilde{y}, \varepsilon) = m(y, \varepsilon), \quad \tilde{f}(\tilde{x}, \tilde{y}) = f(x, y), \quad \tilde{\imath}(\tilde{x}, \tilde{y}) = \sigma\, i(x, y). \tag{A17}$$

To prove the statement of the proposition, it suffices to show that $\{\tilde{f}(\tilde{x}, \tilde{y}), \tilde{\imath}(\tilde{x}, \tilde{y})\}$ is the unique solution to

$$\tilde{f}(\tilde{x}, \tilde{y}) = \min\left\{\max\left\{\mathrm{e}^{-\delta}\,\mathbb{E}_{\tilde{y}}\,\hat{\tilde{M}} \tilde{f}[\mathrm{e}^{-\delta}\tilde{\imath}(\tilde{x}, \tilde{y}) + \hat{\tilde{Y}}, \hat{\tilde{Y}}], \tilde{p}(\tilde{x})\right\} - k, \tilde{p}(\tilde{b})\right\} \tag{A18}$$

$$\tilde{\imath}(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{x} - \tilde{p}^{-1}[\tilde{f}(\tilde{x}, \tilde{y})], & \tilde{x} > x_{\tilde{f}}(\tilde{y}) \\ x_{\tilde{f}}(\tilde{y}) - \tilde{p}^{-1}(0), & \tilde{x} \leq x_{\tilde{f}}(\tilde{y}) \end{cases} \tag{A19}$$

[26]Condition (A13) obviously holds if, for example, $\{Y_t\}$ is iid and follows a truncated normal distribution with mean μ_0, variance σ_0^2, and truncation thresholds $y_l < y_u$. Because in this case, $\{\tilde{Y}_t\}$ is iid and follows a truncated normal distribution as well, with mean $\mu + \sigma\mu_0$, variance $\sigma^2\sigma_0^2$, and truncation thresholds $\mu + \sigma y_l < \mu + \sigma y_u$. Note that (A13) does not hold if $\{Y_t\}$ and $\{\tilde{Y}_t\}$ do not follow the same *type* of distribution. For example, it does not hold if $\{Y_t\}$ is iid lognormally distributed, since $\{\tilde{Y}_t\}$ is not lognormally distributed as a linear transform of $\{Y_t\}$.

[27]In general, Z_t is a multivariate Markov process and Y_t corresponds to one dimension of Z_t.

where

$$x_{\tilde{f}}(\tilde{y}) := \inf\left\{\tilde{x} \geq \tilde{p}^{-1}(0) : \tilde{f}(\tilde{x}, \tilde{y}) = 0\right\}.$$

This is true by referring to (A15)–(A16). In particular, by (A14) and (A17),

$$\hat{\tilde{M}} = \tilde{m}(\hat{\tilde{Y}}, \hat{\varepsilon}) = m(\hat{Y}, \hat{\varepsilon}) = \hat{M}, \quad \tilde{p}(\tilde{x}) = p(x) \quad \text{and} \quad \tilde{p}(\tilde{b}) = p(b).$$

Furthermore, $\tilde{\Psi}(\tilde{y}, \hat{\tilde{Y}}) = \Psi(y, \hat{Y})$ by the definition in (A13) and

$$\begin{aligned}\tilde{f}\left[\mathrm{e}^{-\delta}\tilde{\imath}(\tilde{x}, \tilde{y}) + \hat{\tilde{Y}}, \hat{\tilde{Y}}\right] &= \tilde{f}\left[\mathrm{e}^{-\delta}\sigma i(x, y) + \mu + \sigma\hat{Y}, \hat{\tilde{Y}}\right] \\ &= \tilde{f}\left[\mu + \sigma\left(\mathrm{e}^{-\delta} i(x, y) + \hat{Y}\right), \hat{\tilde{Y}}\right] = f\left[\mathrm{e}^{-\delta} i(x, y) + \hat{Y}, \hat{Y}\right].\end{aligned}$$

The above analysis implies that (A18) holds. To see that (A19) holds, note that

$$\begin{aligned}x_{\tilde{f}}(\tilde{y}) &= \inf\left\{\mu + \sigma x \geq \mu - \frac{a\sigma}{d} : f(x, y) = 0\right\} \\ &= \mu + \sigma \inf\left\{x \geq p^{-1}(0) : f(x, y) = 0\right\} = \mu + \sigma x_f(y),\end{aligned}$$

where we have used the definition of p and $\tilde{p}$. This yields $\tilde{x} < x_{\tilde{f}}(\tilde{y})$ iff $x < x_f(y)$. In combination with (A16), we obtain

$$\tilde{\imath}(\tilde{x}, \tilde{y}) = \sigma i(x, y) = \begin{cases}\sigma\left(x - p^{-1}[f(x, y)]\right), & \tilde{x} < x_{\tilde{f}}(\tilde{y}), \\ \sigma\left(x_f(y) - p^{-1}(0)\right), & \tilde{x} \geq x_{\tilde{f}}(\tilde{y}).\end{cases}$$

When $\tilde{x} < x_{\tilde{f}}(\tilde{y})$, using (A17) and the definition of p and $\tilde{p}$, we obtain

$$\begin{aligned}\sigma\left(x - p^{-1}[f(x, y)]\right) &= \sigma x - \sigma p^{-1}[\tilde{f}(\tilde{x}, \tilde{y})] \\ &= \tilde{x} - \mu - \sigma\left(\frac{\tilde{f}(\tilde{x}, \tilde{y}) - a}{d}\right) = \tilde{x} - \tilde{p}^{-1}[\tilde{f}(\tilde{x}, \tilde{y})].\end{aligned}$$

When $\tilde{x} \geq x_{\tilde{f}}(\tilde{y})$, using the definition of p and $\tilde{p}$ again yields

$$\sigma\left(x_f(y) - p^{-1}(0)\right) = x_{\tilde{f}}(\tilde{y}) - \mu + \frac{a\sigma}{d} = x_{\tilde{f}}(\tilde{y}) - \tilde{p}^{-1}(0).$$

The above analysis implies that (A19) holds. Therefore, economies E and $\tilde{E}$ generate the same commodity price process. □

E. Algorithms

The storage model is solved by a modified version of the endogenous grid method of Carroll (2006). The candidate space $\mathscr{C}$ and $\bar{p}_f(z)$ are as defined in Appendix A. We derive the following property in order to handle free-disposal and state-dependent discounting in the numerical computation.

Lemma E.1. *For each f in the candidate space $\mathscr{C}$, we have $x > p^{-1}[\bar{p}_f(z)]$ if and only if*

$$Tf(x, z) = \max\left\{\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M} f\left(\mathrm{e}^{-\delta}\left(x - p^{-1}[Tf(x, z)]\right) + \hat{Y}, \hat{Z}\right) - k, 0\right\}$$

Proof. Immediate by Lemma A.5 (ii)–(iii) and the fact that $Tf(x, z)$ is decreasing in x. □

E.1. The Endogenous Grid Algorithm

To simplify exposition, we consider the setting of Section 4, but it is clear that our method can be easily adjusted to analyze more general settings (e.g., state-dependent production, and correlated interest rate and production processes). We discretize the interest rate process into a finite Markov Chain following Tauchen (1986). The states of the gross interest rates are indexed by j and m, and the transition matrix has elements $\Phi_{j,m}$. Moreover, we use $\mathcal{D}(X, R) = p^{-1}[f(X, R)]$ to denote a candidate equilibrium demand function. The endogenous grid algorithm for computing the equilibrium pricing rule is described in Algorithm 1.

Algorithm 1 The endogenous grid algorithm

Step 1. Initialization step. Choose a convergence criterion $\varpi > 0$, a grid on storage $\{I_k\}$ starting at 0, a grid on production shocks for numerical integration $\{Y_n\}$ with associated weights w_n, and an initial policy rule (guessed): $\{X^1_{k,j}\}$ and $\{P^1_{k,j}\}$. Start iteration at $i = 1$.

Step 2. Update the demand function via interpolation and extrapolation:

$$p^{-1}\left(P^i_{k,j}\right) = \mathcal{D}^i\left(X^i_{k,j}, R_j\right). \tag{A20}$$

Step 3. Obtain prices and availability consistent with the grid of stocks and interest rates:

$$P^{i+1}_{k,j} = \max\left\{ \mathrm{e}^{-\delta} \sum_{n,m} w_n \Phi_{j,m} p\left(\mathcal{D}^i\left(Y_n + \mathrm{e}^{-\delta} I_k, R_m\right)\right) \Big/ R_m,\ 0 \right\}, \tag{A21}$$

$$X^{i+1}_{k,j} = I_k + p^{-1}\left(P^{i+1}_{k,j}\right). \tag{A22}$$

Step 4. Terminal step. If $\max|P^{i+1}_{k,j} - P^i_{k,j}| \geq \varpi$ then increment i to $i + 1$ and go to step 2. Otherwise, approximate the equilibrium pricing rule by $f^*(X, R) = p[\mathcal{D}^i(X, R)]$.

In particular, we choose to approximate the demand function $\mathcal{D}(X, R)$ in Step 2 instead of the price function $f(X, R)$. This is helpful for improving both precision and stability of the algorithm when the demand function diverges at the lower bound of the endogenous state space. A typical example is the exponential demand $p(x) = x^{-1/\lambda} (\lambda > 0)$, which is commonly adopted by applied research (see, e.g., Deaton and Laroque, 1992; Gouel and Legrand, 2022). If the inverse demand function is linear as in Section 4, however, then it is innocuous to approximate the price function directly.

Moreover, the validity and convergence of the updating process in Step 3 are justified by Theorem A.1, Lemma A.5, and Lemma E.1 above.

E.2. Solution Precision

To evaluate the precision of the numerical solution, we refer to a suitably adjusted version of the bounded rationality measure originally designed by Judd (1992), which we name as the Euler equation error and measures how much solutions violate the optimization conditions. In the current context, it is defined at state (x, z) as

$$EE_f(x, z) = 1 - \frac{\mathcal{D}_1(x, z)}{\mathcal{D}_2(x, z)},$$

where f is the numerical solution of the equilibrium price,

$$\mathcal{D}_1(x, z) = p^{-1}\left[\min\left\{\max\left\{\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}f(\hat{X}, \hat{Z}) - k, p(x)\right\}, p(b)\right\}\right] - b$$

and $\mathcal{D}_2(x, z) = p^{-1}[f(x, z)] - b$. In particular, both $\mathcal{D}_1(x, z)$ and $\mathcal{D}_2(x, z)$ are expressed in terms of the *relative* demand for commodity, since b is the greatest lower bound (hence corresponds to the *zero* level) of the total available supply. Therefore, $EE_f(x, z)$ measures the error at state (x, z), in terms of the quantity consumed, incurred by using the numerical solution instead of the true equilibrium pricing rule.

To evaluate the precision of the endogenous grid algorithm in the current context, we simulate a time series $\{(X_t, R_t)\}_{t=1}^{T}$ of length T =20,000 based on the state evolution path $X_{t+1} = \mathrm{e}^{-\delta} i(X_t, R_t) + Y_t$ and $R_{t+1} \sim \Phi(R_t, \cdot)$, where (X_0, R_0) is given, and

$$i(X_t, R_t) = \min\{X_t, x_f^*(R_t)\} - p^{-1}[f(X_t, R_t)]$$

is the equilibrium storage function computed by the endogenous grid algorithm. We discard the first 1,000 draws, and then compute the Euler equation error at the truncated time series. When applying the endogenous grid algorithm, we use an exponential grid for storage in the range $[0, 2]$ with median value 0.5, function iteration is implemented via linear interpolation and linear extrapolation, and we terminate the iteration process at precision $\varpi = 10^{-4}$. The rest of the setting is same to Section 4.

Summary statistics (maximum as well as 95-th percentile) are reported in Table A1, where K is the number of grid points for storage, N is the number of state points for interest rates, and precision at (x, z) is evaluated as $\log_{10}|EE_f(x, z)|$. The results demonstrate that the endogenous grid algorithm attains a high level of precision, with an Euler equation error uniformly less than 0.025% ($\max|EE_f| = 10^{-3.64} \approx 0.00023$).

E.3. The Generalized Impulse Response Function

To properly capture the nonlinear asymmetric dynamics of the competitive storage model and effectively study the dynamic causal effect of interest rates on commodity prices, we refer to the generalized impulse response function proposed by Koop et al. (1996), which defines IRFs as state-and-history-dependent random variables and is applicable to both linear and nonlinear multivariate models. We are interested in calculating the IRFs when (X_{t-1}, R_{t-1}) are held at different percentiles of the stationary distribution.

Table A1 – Precision under different grid sizes and different parameters

A. Different grid sizes

	$K = 100$			$K = 200$			$K = 1{,}000$		
Precision	$N = 7$	$N = 51$	$N = 101$	$N = 7$	$N = 51$	$N = 101$	$N = 7$	$N = 51$	$N = 101$
max	−3.68	−3.64	−3.64	−3.98	−4.02	−4.02	−4.68	−5.21	−5.18
95%	−4.66	−4.66	−4.67	−5.21	−5.39	−5.39	−7.02	−6.74	−6.75

B. Different parameters

	$\lambda = -0.03$			$\lambda = -0.06$			$\lambda = -0.15$		
Precision	$\delta = 0.01$	$\delta = 0.02$	$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.02$	$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.02$	$\delta = 0.05$
max	−3.63	−3.66	−3.64	−3.64	−3.64	−3.59	−3.68	−3.68	−3.69
95%	−5.09	−4.90	−4.65	−4.90	−4.66	−4.45	−4.63	−4.52	−4.97

Notes: In Panel A, we fix $\lambda = -0.06$ and $\delta = 0.02$, simulate a time series of length $T = 20{,}000$, discard the first 1,000 draws, and then compute the level of precision as $\log_{10}|EE_f|$. When applying the endogenous grid algorithm, we use an exponential grid for storage in the range $[0, 2]$ with median value 0.5, function iteration is implemented via linear interpolation and linear extrapolation, and we terminate the iteration process at precision $\varpi = 10^{-4}$. The rest of the setting is same to Section 4. In Panel B, we fix the grid size to $K = 100$ and $N = 51$, and vary the parameters.

Algorithm 2 clarifies the computation process of the generalized IRFs based on the setting of Section 4. However, the algorithm can be easily extended to handle more general settings as formulated in Section 2, where more advanced interest rate and production setups are allowed. To proceed, we define

$$F(x, R, Y) := \mathrm{e}^{-\delta}\left(\min\{x, x^*(R)\} - p^{-1}[f^*(x, R)]\right) + Y.$$

The stationary distribution of the state process is computed based on ergodicity. Once f^*, i^*, and x^* are calculated, we simulate a time series of $\{(X_t, R_t)\}_{t=1}^{T}$ according to

$$X_{t+1} = \mathrm{e}^{-\delta}\left(\min\{X_t, x^*(R_t)\} - p^{-1}[f^*(X_t, R_t)]\right) + Y_{t+1} \quad \text{and} \quad R_{t+1} \sim \Pi(R_t, \cdot)$$

for $T = 200{,}000$ periods, discard the first 50,000 samples, and use the remainder to approximate the stationary distribution.

Algorithm 2 The generalized impulse response function

Step 1. Initialization step. Choose initial values for X_{t-1} and R_{t-1}, and a finite horizon H and a size of Monte Carlo samples S. Furthermore, set the initial samples as

$$\tilde{X}^s_{t-1} = X^s_{t-1} \equiv X_{t-1}, \quad R^s_{t-1} = \tilde{R}^s_{t-1} \equiv R_{t-1} \quad \text{and} \quad \tilde{R}^s_t \equiv \mu_R + \rho_R \tilde{R}^s_{t-1} + \sigma_R.$$

Step 2. Randomly sample $(H+1) \times S$ values of production shocks $\left\{Y^s_{t+h}\right\}^{(H,S)}_{(h,s)=(0,1)}$.

Step 3. (Baseline Economy) Sample $(H+1) \times S$ values of interest rate

$$\left\{R^s_{t+h}\right\}^{(H,S)}_{(h,s)=(0,1)} \quad \text{where} \;\; R^s_{t+h} \sim \Pi(R^s_{t+h-1}, \cdot).$$

Step 4. (Impulse Shock Economy) Sample $H \times S$ values of interest rate

$$\left\{\tilde{R}^s_{t+h}\right\}^{(H,S)}_{(h,s)=(1,1)} \quad \text{where} \;\; \tilde{R}^s_{t+h} \sim \Pi(\tilde{R}^s_{t+h-1}, \cdot).$$

Step 5. For $h = 0, \dots, H$ and $s = 1, \dots, S$, compute the sequence of availability

$$X^s_{t+h} = F(X^s_{t+h-1}, R^s_{t+h-1}, Y^s_{t+h}) \quad \text{and} \quad \tilde{X}^s_{t+h} = F(\tilde{X}^s_{t+h-1}, \tilde{R}^s_{t+h-1}, Y^s_{t+h}).$$

Step 6. For $h = 0, \dots, H$, compute the period-$(t+h)$ impulse response

$$IRF(t+h) = \frac{1}{S}\sum_{s=1}^{S} f^*(\tilde{X}^s_{t+h}, \tilde{R}^s_{t+h}) - \frac{1}{S}\sum_{s=1}^{S} f^*(X^s_{t+h}, R^s_{t+h}).$$